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SHANGHAI JIAO TONG UNIVERSITY

上海交通大学
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CS 445: Combinatorics

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Basics

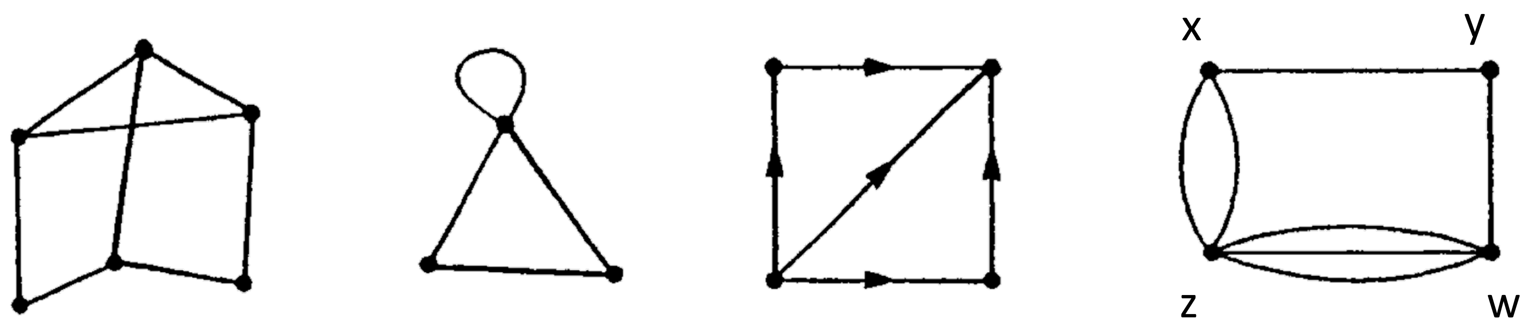
Graphs

- Definition A graph G is a pair (V, E)
  - V: set of vertices
  - E: set of edges
  - $e \in E$ corresponds to a pair of endpoints $x, y \in V$

We mainly focus on
Simple graph:
No loops, no multi-edges

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

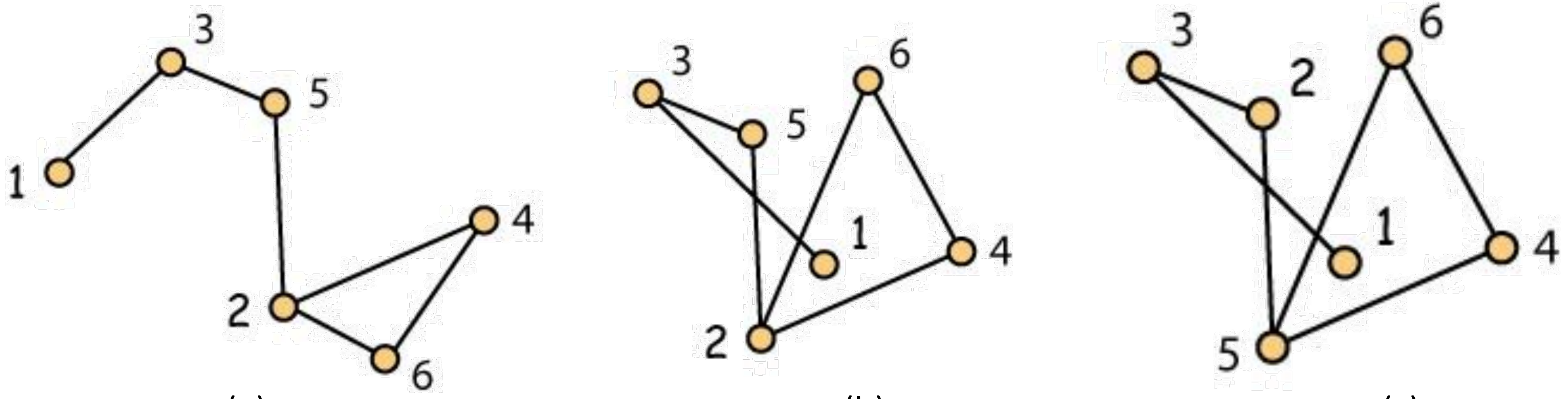


(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

Figure 1.2

Graphs: All about adjacency

- Same graph or not



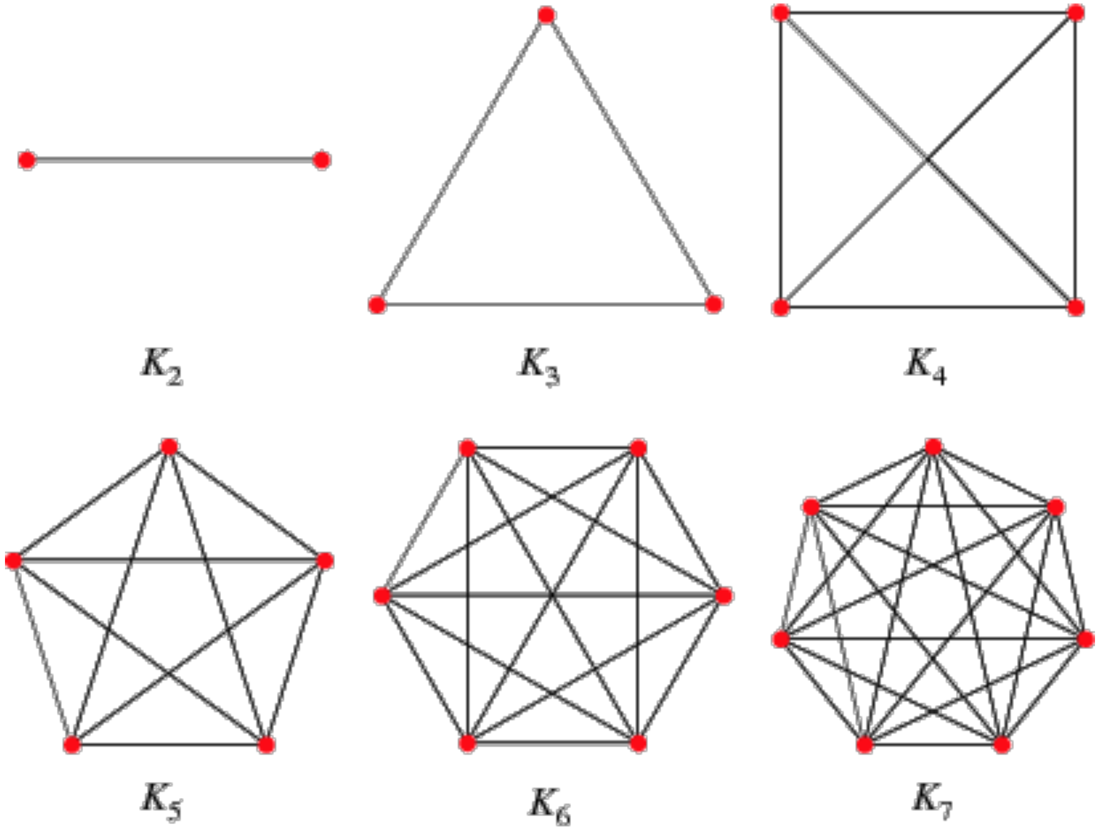
(a) (b) (c)

- Two graphs $G_1 = (V_1, E_1), G_1 = (V_2, E_2)$ are isomorphic if there is a bijection $f\colon V_1 \to V_2$ s.t.

$$e = \{a, b\} \in E_1 \Leftrightarrow f(e) := \{f(a), f(b)\} \in E_2$$

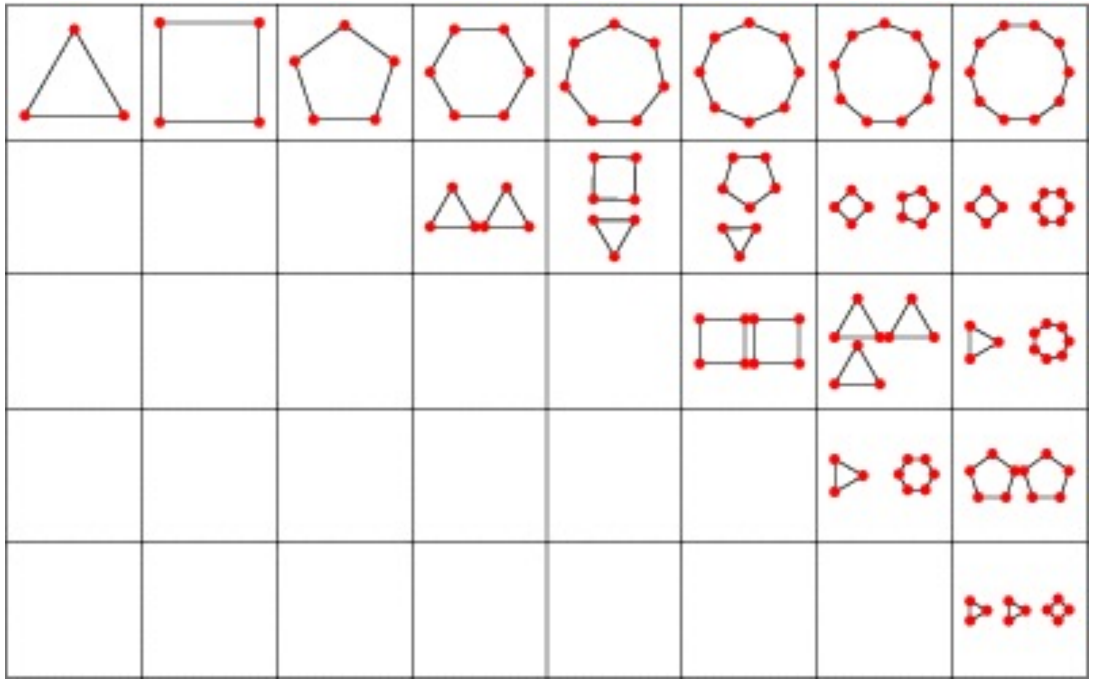
Example: Complete graphs

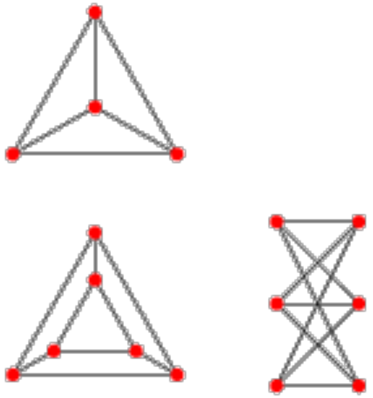
- There is an edge between every pair of vertices

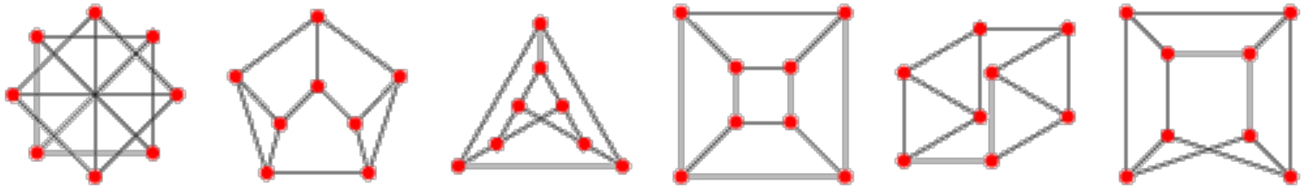


Example: Regular graphs

- Every vertex has the same degree

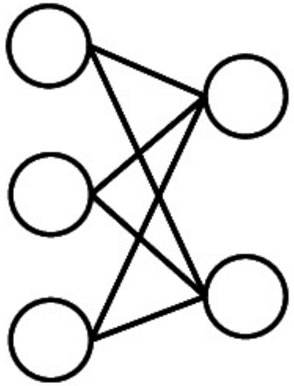




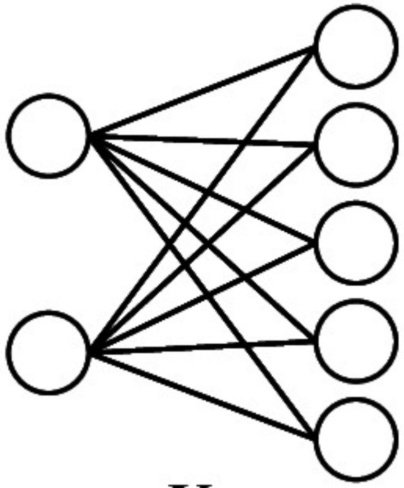


Example: Bipartite graphs

- The vertex set can be partitioned into two sets X and Y such that every edge in G has one end vertex in X and the other in Y
- Complete bipartite graphs



$\mathbf{K_{3,2}}$



$\mathbf{K_{2,5}}$

Example (1A, L): Peterson graph

- Show that the following two graphs are same/isomorphic

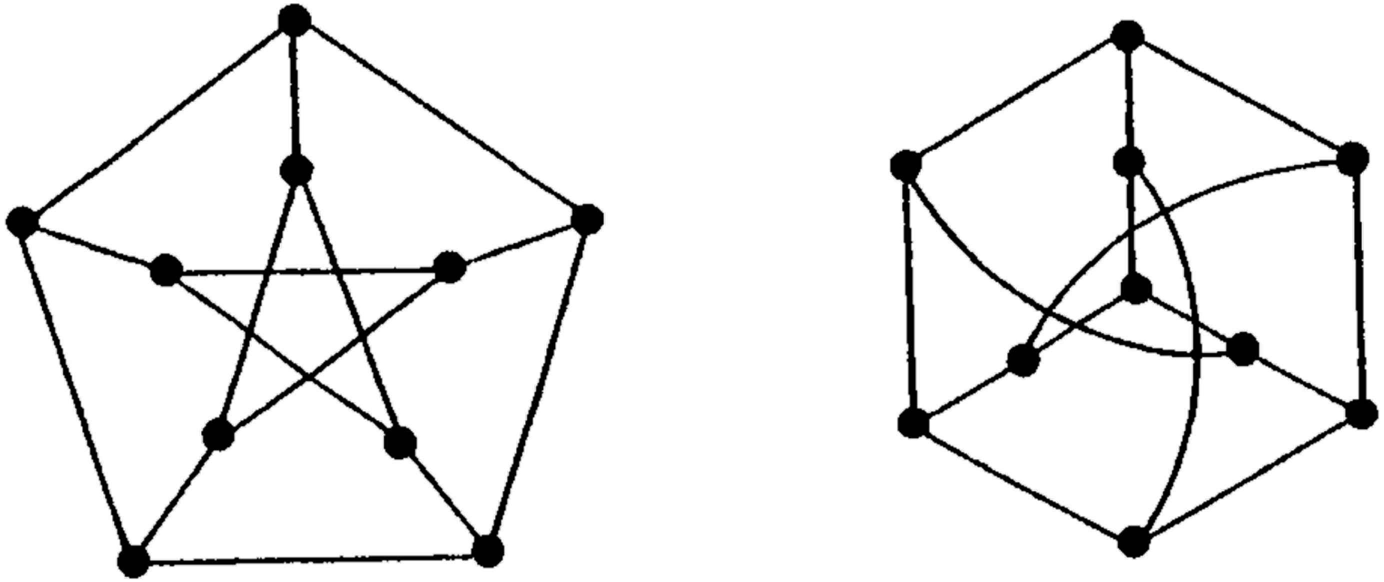
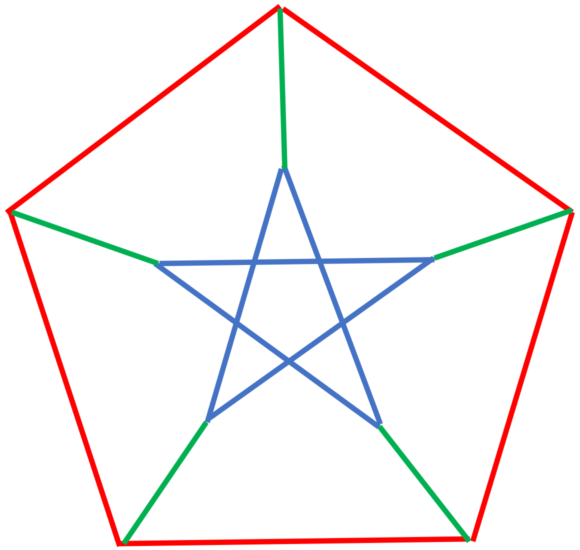
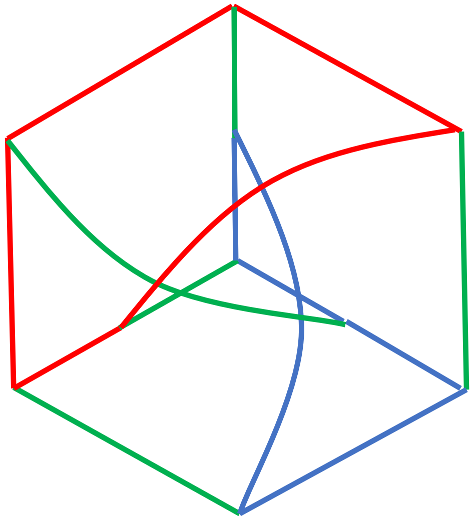


Figure 1.4

Example: Peterson graph (cont.)

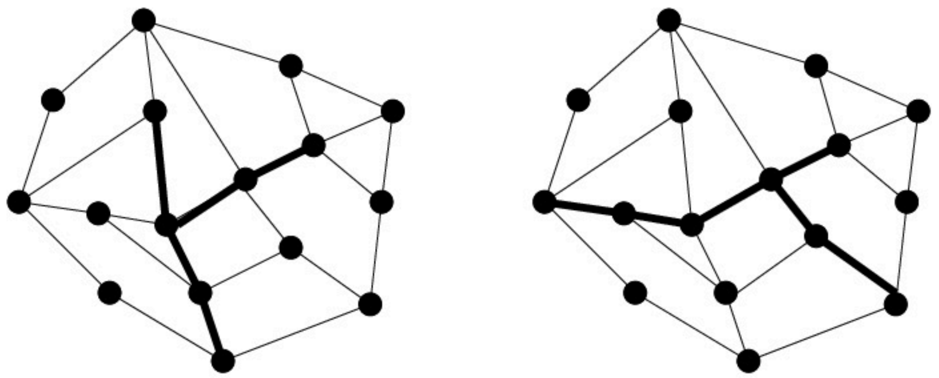
- Show that the following two graphs are same/isomorphic



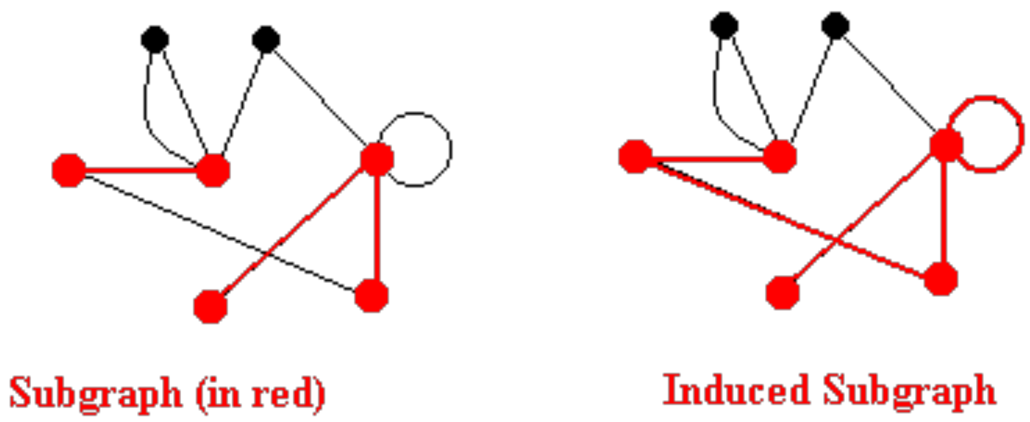


Subgraphs

- A subgraph of a graph G is a graph H such that
$$V(H) \subseteq V(G), E(H) \subseteq E(G)$$
and the ends of an edge $e \in E(H)$ are the same as its ends in G
  - H is a spanning subgraph when $V(H) = V(G)$
  - The subgraph of G induced by a subset $S \subseteq V(G)$ is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S

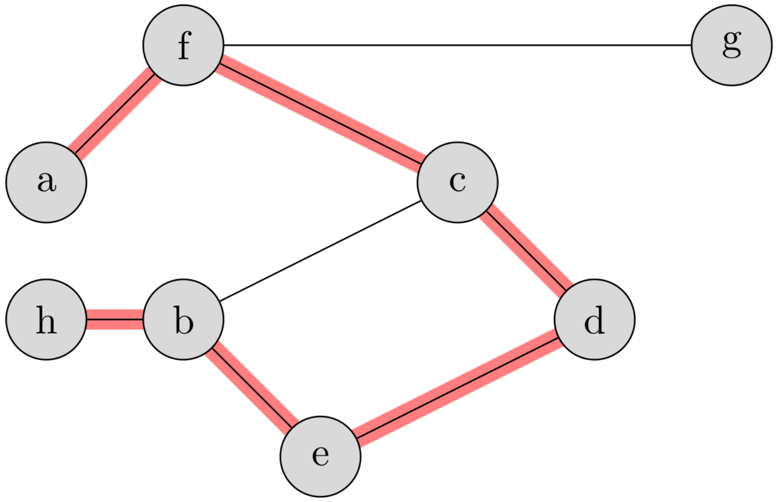


(a) (b)



Paths (路径)

- A path is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$ where vertices are all distinct
  - Or it can be written as $v_0 v_1 \dots v_k$ in simple graphs
- P^k: path of length k (the number of edges)

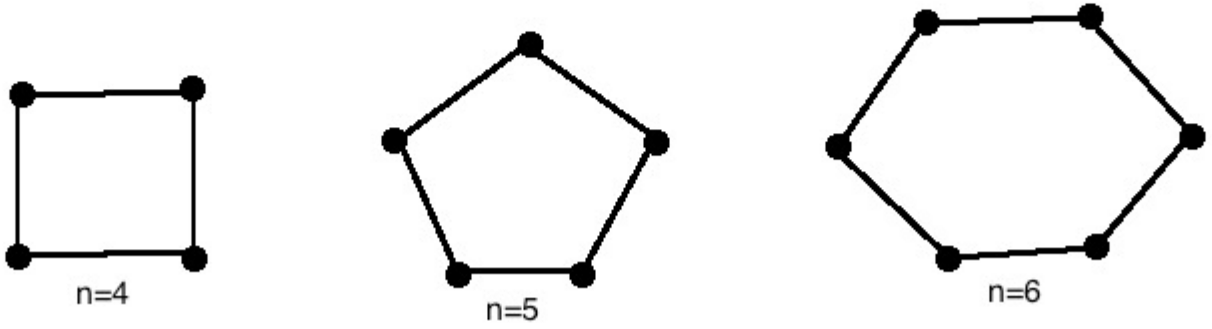


Walk (游走)

- A walk is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$
  - The vertices not necessarily distinct
  - The length = the number of edges
- Proposition (1.2.5, W) Every u-v walk contains a u-v path

Cycles (环)

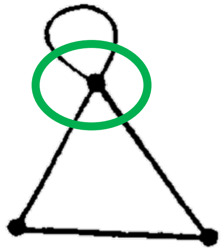
- If $P = x_0x_1 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C \coloneqq P + x_{k-1}x_0$ is called a cycle
- C^k: cycle of length k (the number of edges/vertices)



- Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Neighbors and degree

- Two vertices $a \neq b$ are called adjacent if they are joined by an edge
  - $N(x)$: set of all vertices adjacent to x
    - neighbors of x
  - A vertex is isolated vertex if it has no neighbors
- The number of edges incident with a vertex x is called the degree of x
  - A loop contributes 2 to the degree

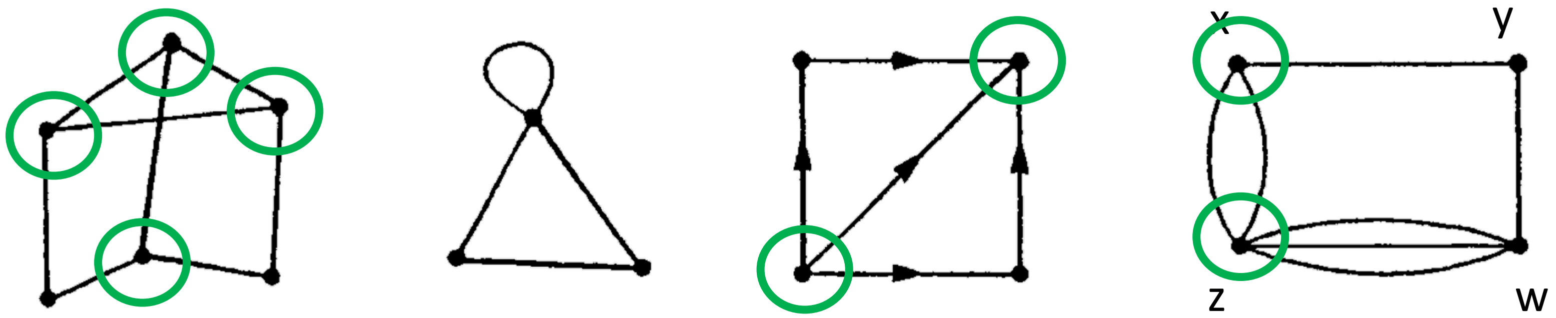


graph with loop

- A graph is finite when both $E(G)$ and $V(G)$ are finite sets

Handshaking Theorem (Euler 1736)

- Theorem A finite graph G has an even number of vertices with odd degree



(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

Figure 1.2

Proof

- Theorem A finite graph G has an even number of vertices with odd degree.
- Proof The degree of x is the number of times it appears in the right column. Thus

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

Degree

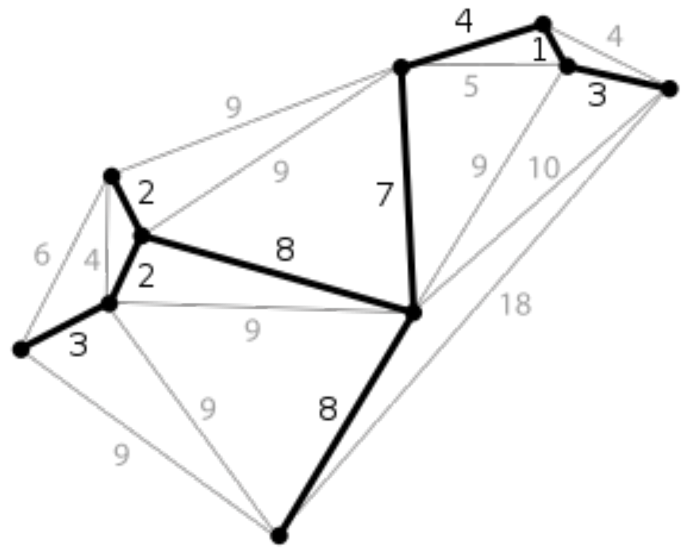
- Minimal degree of G: $\delta(G) = \min\{d(v): v \in V\}$
- Maximal degree of G: $\Delta(G) = \max\{d(v): v \in V\}$
- Average degree of G: $d(G) = \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{2|E|}{|V|}$

- All measure the `density' of a graph

- $d(G) \geq \delta(G)$

Minimal degree guarantees long paths and cycles

- Proposition (1.3.1, D) Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$, provided $\delta(G) \geq 2$.



Distance and diameter

- The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x{\sim}y$ path
  - if no such path exists, we set $d(x, y) \coloneqq \infty$
- The greatest distance between any two vertices in G is the diameter of G

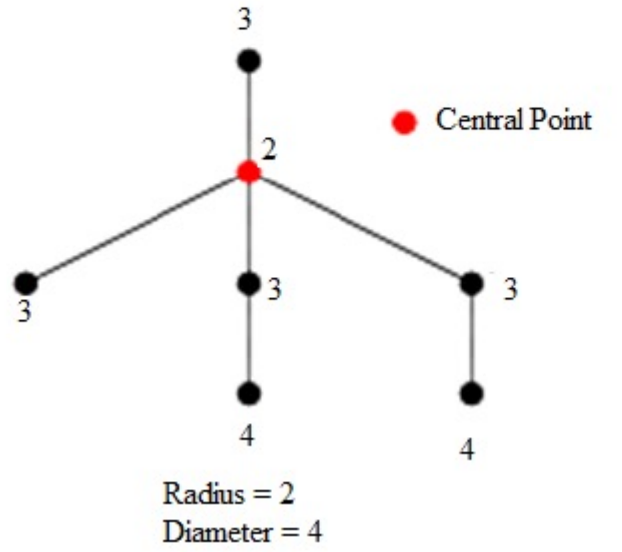
$$\operatorname{diam}(G) = \max_{x,y \in V} d(x, y)$$

Radius and diameter

- A vertex is central in G if its greatest distance from other vertex is smallest, such greatest distance is the radius of G

$$\mathrm{rad(G)} := \min_{x \in V} \max_{y \in V} d(x, y)$$

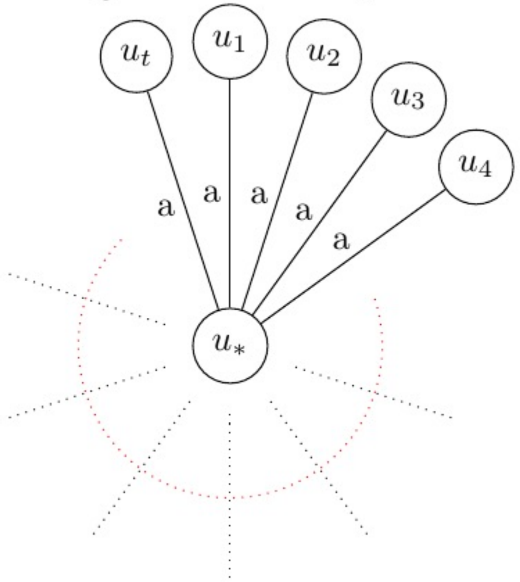
- Proposition (1.4, H; Ex1.6, D) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$



Radius and maximum degree control graph size

- Proposition (1.3.3, D) A graph G with radius at most r and maximum degree at most $\Delta \geq 3$ has fewer than $\frac{\Delta}{\Delta-2}(\Delta-1)^r$.

Figure 1: Star Graph



Lecture 2: Girth, Connectivity and Bipartite Graphs

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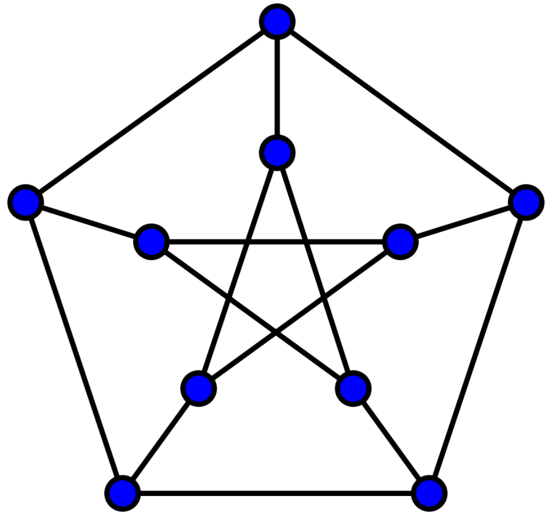
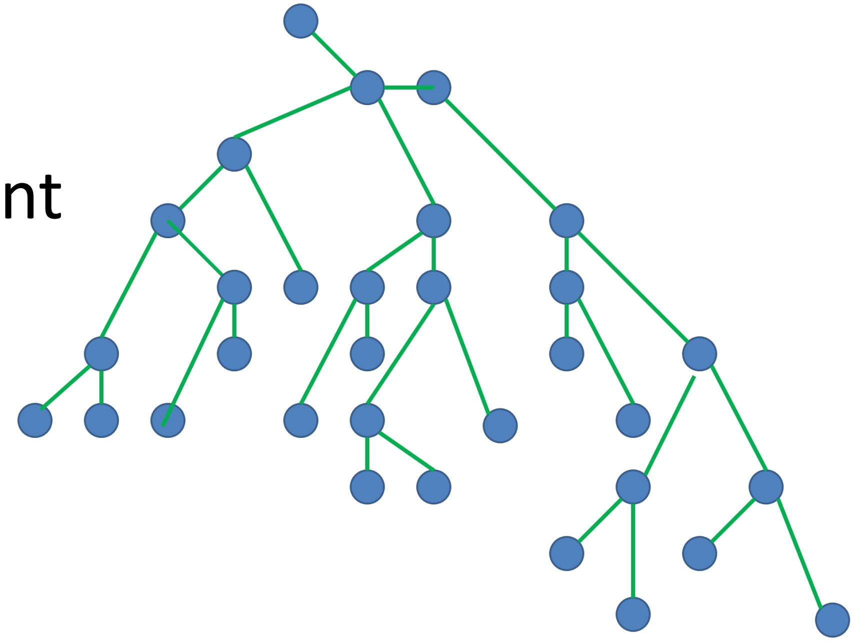
Girth

- The minimum length of a cycle in a graph G is the girth $g(G)$ of G
- Example: The Peterson graph is the unique 5-cage
  - cubic graph (every vertex has degree 3)
  - girth = 5
  - smallest graph satisfies the above properties



Girth (cont.)

- A tree has girth ∞
- Note that a tree can be colored with two different colors
- $\Longrightarrow$ A graph with large girth has small chromatic number?
- Unfortunately NO!
- Theorem (Erdős, 1959) For all k, l, there exists a graph G with $g(G) > l$ and $\chi(G) > k$



Girth and diameter

- Proposition (1.3.2, D) Every graph G containing a cycle satisfies $g(G) \leq 2\,\mathrm{diam}(G) + 1$

- When the equality holds?

Girth and minimal degree lower bounds graph size

- $n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r \text{ is even} \end{cases}$
- Exercise (Ex7, ch1, D) Let G be a graph. If $\delta(G) \geq \delta \geq 2$ and $g(G) \geq g$, then $|G| \geq n_0(\delta, g)$
- Corollary (1.3.5, D) If $\delta(G) \geq 3$, then $g(G) < 2 \log_2 |G|$

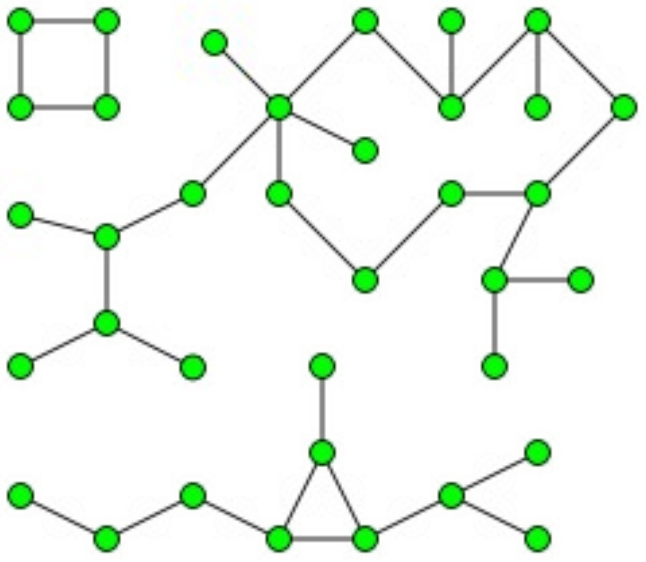
Triangle-free upper bounds # of edges

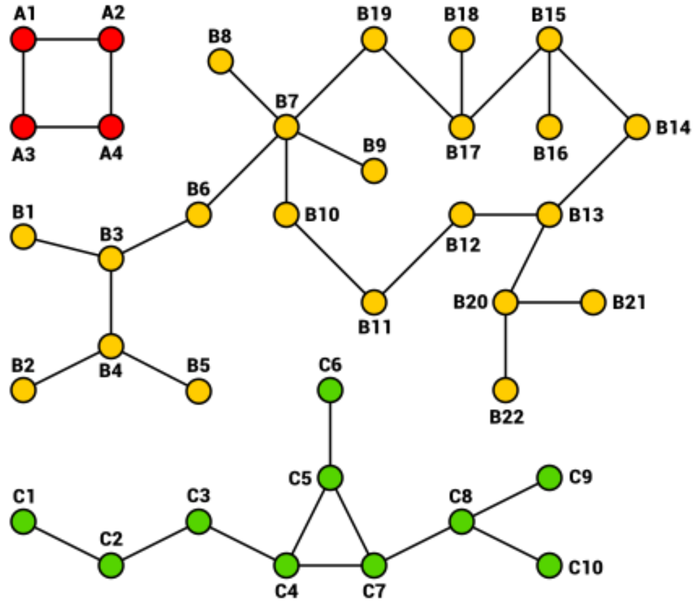
- Theorem (1.3.23, W, Mantel 1907) The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$

- The bound is best possible
- There is a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$
- Extremal problems

Connected, connected component

- A graph G is connected if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of G is a (connected) component





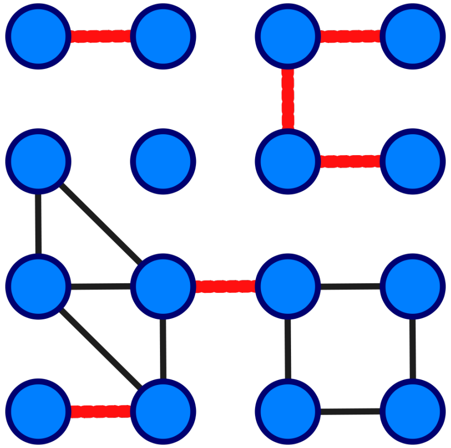
Quiz

- Problem (1B, L) Suppose G is a graph on 10 vertices that is not connected. Prove that G has at most 36 edges. Can equality occur?
- More general (Ex9, S1.1.2, H) Let G be a graph of order n that is not connected. What is the maximum size of G?

Connected vs. minimal degree

- Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected
- (Ex16, S1.1.2, H; 1.3.16, W)
  If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected

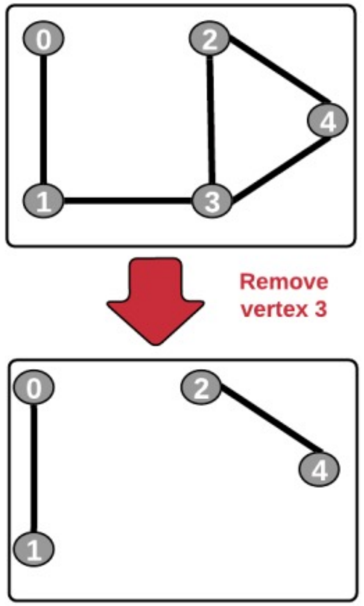
- Extremal problems
- “best possible” “sharp”



Add/delete an edge

- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
  - $\Rightarrow$ deleting an edge increases the number of components by 0 or 1
- Proposition (1.2.11, W)
  Every graph with n vertices and k edges has at least $n - k$ components
- An edge e is called a bridge if the graph $G - e$ has more components
- Proposition (1.2.14, W)
  An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
  - Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

Cut vertex and connectivity

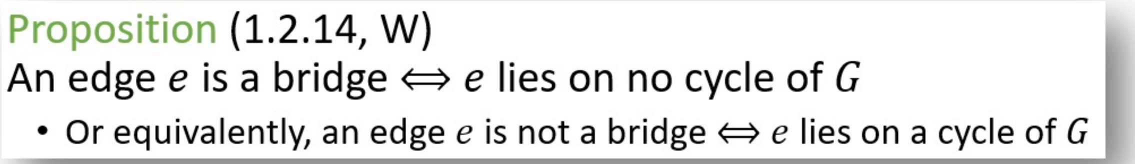


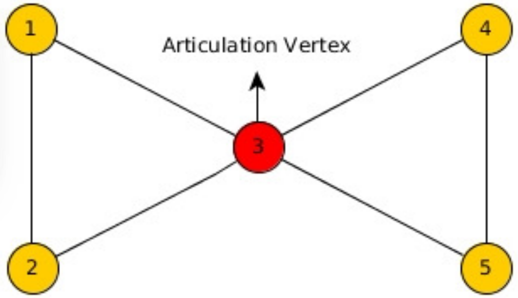
- A node v is a cut vertex if the graph $G - v$ has more components
- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected, or trivial (a graph of order 0 or 1)
- The connectivity, $\kappa(G)$, is the minimum size of a cut set of G
  - The graph is k-connected for any $k \leq \kappa(G)$

Connectivity properties

- $\kappa(K^n) = n - 1$
- If G is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n, then
$$1 \leq \kappa(G) \leq n - 2$$

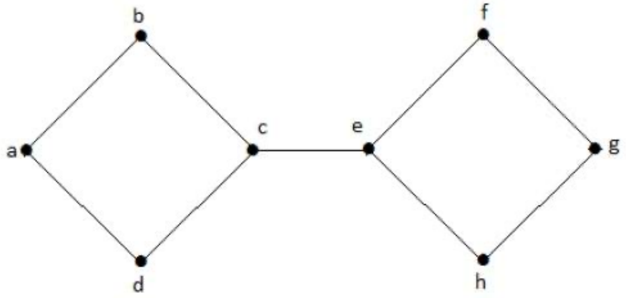
Connectivity properties (cont.)





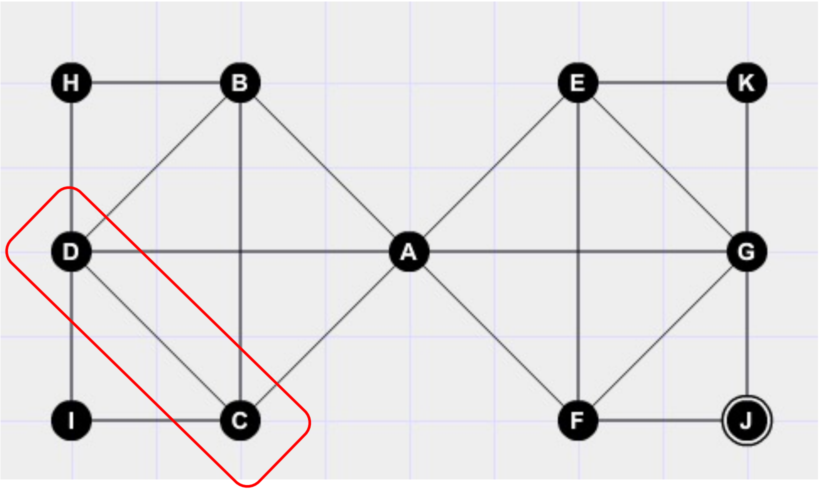
- $\kappa(G) \geq 2 \Leftrightarrow G$ is connected and has no cut vertices
- A vertex lies on a cycle $\not\Rightarrow$ it is not a cut vertex
  - $\Rightarrow$ (Ex13, S1.1.2, H) Every vertex of a connected graph G lies on at least one cycle $\not\Rightarrow \kappa(G) \geq 2$
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

- (Ex12, S1.1.2, H) G has a cut vertex vs. G has a bridge



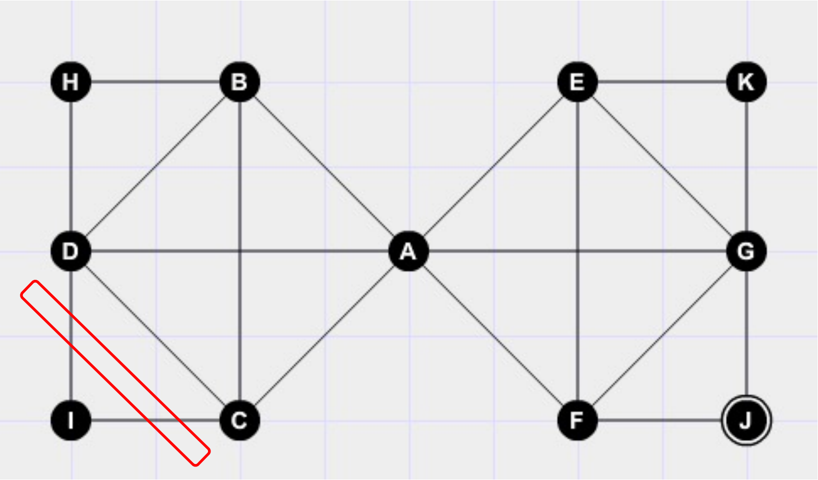
Connectivity and minimal degree

- (Ex15, S1.1.2, H)
- $\kappa(G) \leq \delta(G)$
- If $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$

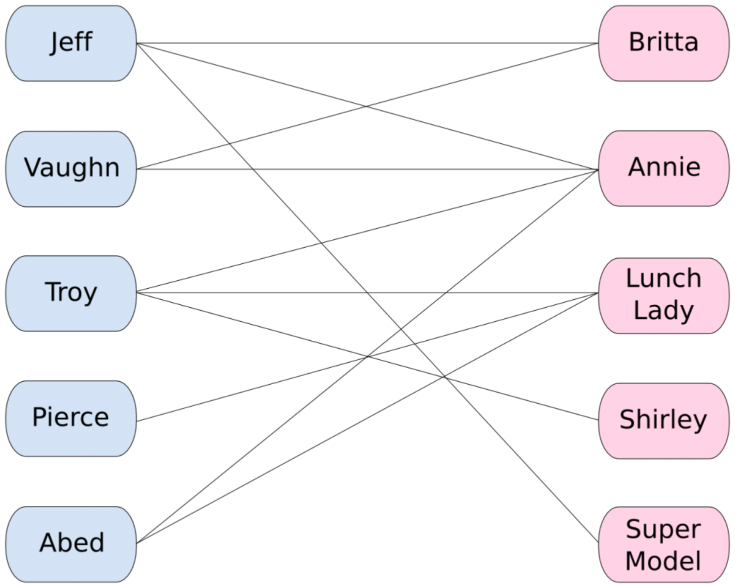


Edge-connectivity

- A proper subset $F \subset E$ is edge cut set if the graph $G - F$ is disconnected
- The edge-connectivity $\lambda(G)$ is the minimal size of edge cut set
- $\lambda(G) = 0$ if G is disconnected
- Proposition (1.4.2, D) If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$



Bipartite graphs



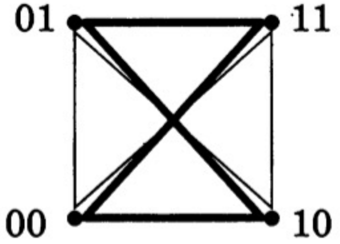
- Theorem (1.2.18, W, Kőnig 1936)
  A graph is bipartite $\Longleftrightarrow$ it contains no odd cycle

Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Complete graph is a union of bipartite graphs

- The union of graphs $G_1, \dots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$

- Consider an air traffic system with k airlines
  - Each pair of cities has direct service from at least one airline
  - No airline can schedule a cycle through an odd number of cities
  - Then, what is the maximum number of cities in the system?



- Theorem (1.2.23, W) The complete graph K_n can be expressed as the union of k bipartite graphs $\Leftrightarrow n \leq 2^k$

Bipartite subgraph is large

- Theorem (1.3.19, W) Every loopless graph G has a bipartite subgraph with at least $|E|/2$ edges

Lecture 3: Trees

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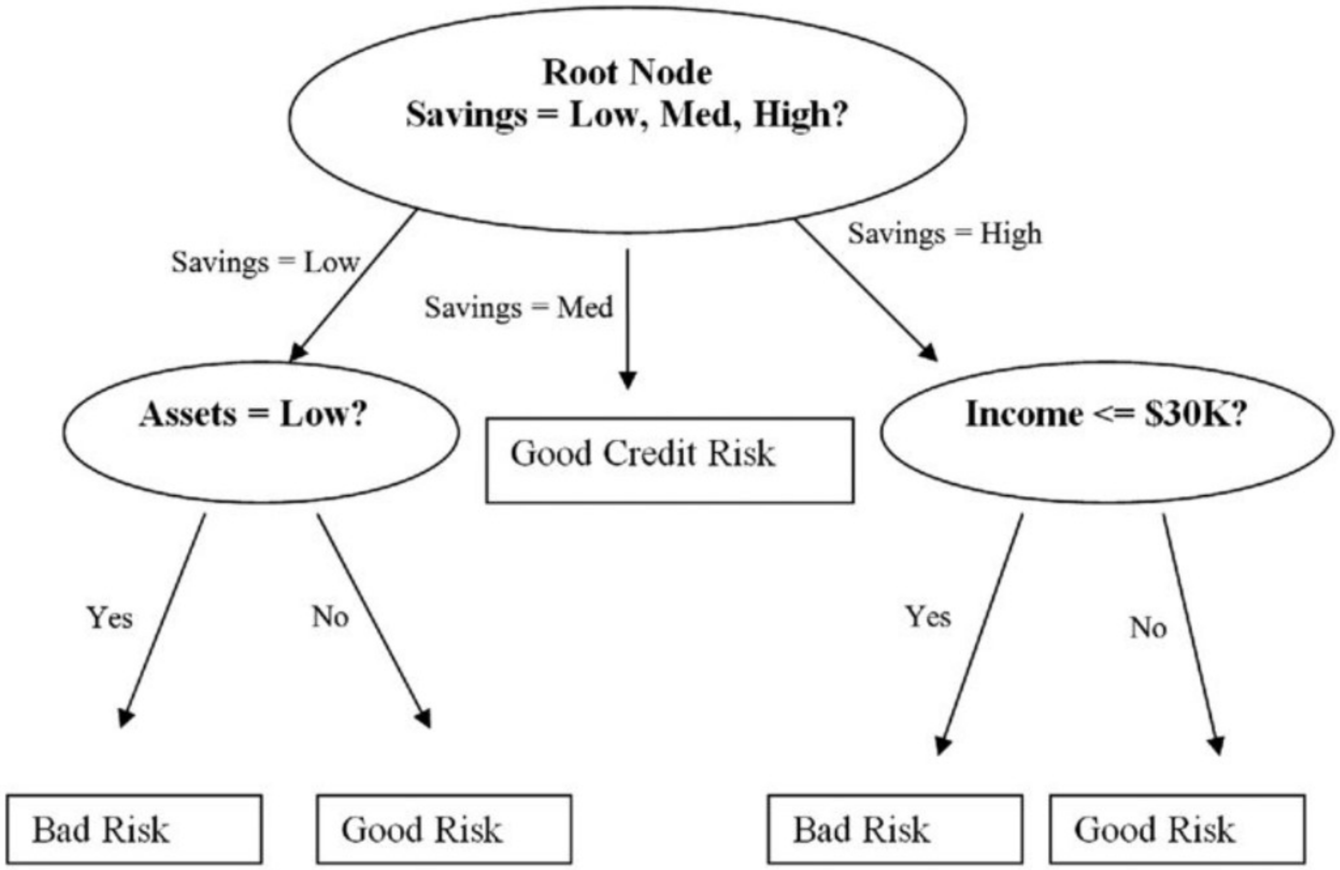
Trees

- A tree is a connected graph T with no cycles



Properties

- Recall that Theorem (1.2.18, W, Kőnig 1936)
  A graph is bipartite $\Leftrightarrow$ it contains no odd cycle
- $\Rightarrow$(Ex 3, S1.3.1, H) A tree of order $n \geq 2$ is a bipartite graph
- Recall that Proposition (1.2.14, W)
  An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
  - Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G
- $\Rightarrow$ Every edge in a tree is a bridge
- T is a tree $\Leftrightarrow$ T is minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$

Equivalent definitions (Theorem 1.5.1, D)

- T is a tree of order n
  $\Leftrightarrow$ Any two vertices of T are linked by a unique path in T
  $\Leftrightarrow$ T is minimally connected
  - i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$

  $\Leftrightarrow$ T is maximally acyclic
  - i.e. T contains no cycle but $T + xy$ does for any non-adjacent vertices $x, y \in T$

  $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n - 1$ edges

  $\Leftrightarrow$ (Theorem 1.13, H) T is acyclic with $n - 1$ edges

Leaves of tree

- A vertex of degree 1 in a tree is called a leaf
- Theorem (1.14, H; Ex9, S1.3.2, H) Let T be a tree of order $n \geq 2$. Then T has at least two leaves
- (Ex3, S1.3.2, H) Let T be a tree with max degree Δ. Then T has at least Δ leaves
- (Ex10, S1.3.2, H) Let T be a tree of order $n \geq 2$. Then the number of leaves is
$$2 + \sum_{v:d(v)\geq 3} (d(v) - 2)$$
- (Ex8, S1.3.2, H) Every nonleaf in a tree is a cut vertex
- Every leaf node is not a cut vertex

The center of a tree is a vertex or ‘an edge’

- Theorem (1.15, H) In any tree, the center is either a single vertex or a pair of adjacent vertices

Any tree can be embedded in a ‘dense’ graph

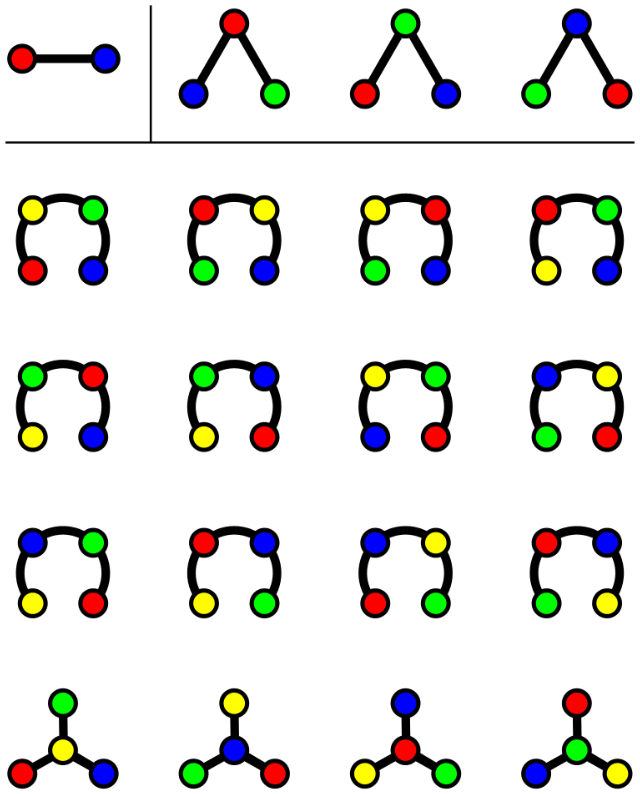
- Theorem (1.16, H) Let T be a tree of order $k + 1$ with k edges. Let G be a graph with $\delta(G) \geq k$. Then G contains T as a subgraph

Spanning tree

- Given a graph G and a subgraph T, T is a spanning tree of G if T is a tree that contains every vertex of G
- Example: A telecommunications company tries to lay cable in a new neighbourhood
- Proposition (2.1.5c, W) Every connected graph contains a spanning tree

Cayley's tree formula

- Theorem (1.18, H; 2.2.3, W). There are n^{n-2} distinct labeled trees of order n



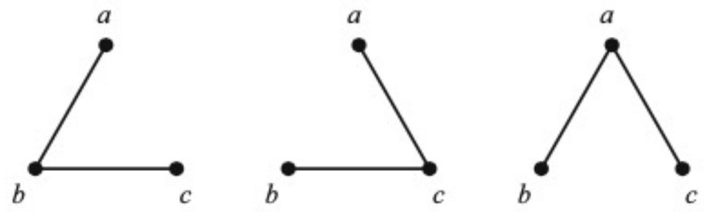


FIGURE 1.45. Labeled trees on three vertices.

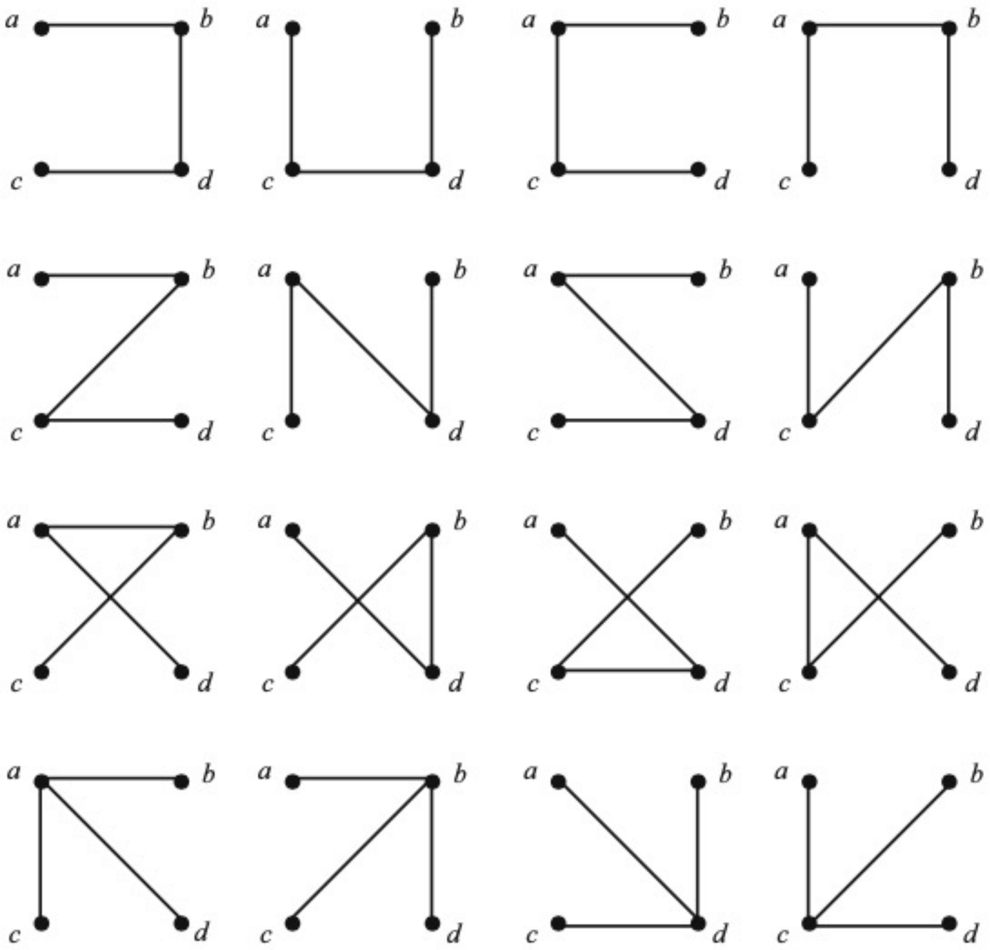


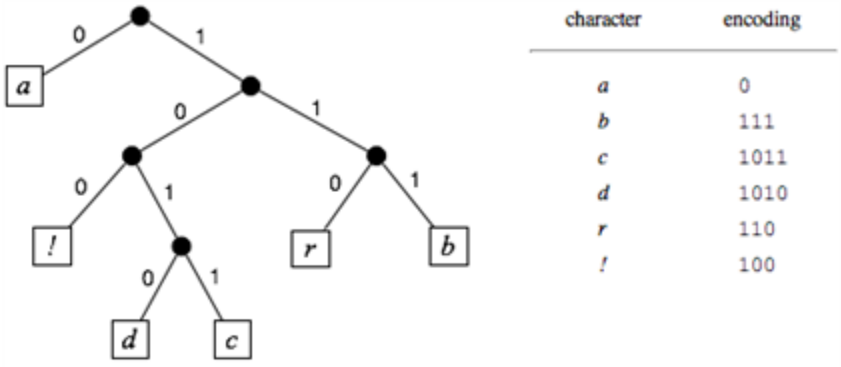
FIGURE 1.46. Labeled trees on four vertices.

Wiener index

- In a communication network, large diameter may be acceptable if most pairs can communicate via short paths. This leads us to study the average distance instead of the maximum
- Wiener index $D(G) = \sum_{u,v \in V(G)} d_G(u, v)$
- Theorem (2.1.14, W) Among trees with n vertices, the Wiener index $D(T)$ is minimized by stars and maximized by paths, both uniquely
- Over all connected n-vertex graphs, $D(G)$ is minimized by K_n and maximized (2.1.16, W) by paths
  - (Lemma 2.1.15, W) If H is a subgraph of G, then $d_G(u, v) \leq d_H(u, v)$

Prefix coding

- A binary tree is a rooted plane tree where each vertex has at most two children
- Given large computer files and limited storage, we want to encode characters as binary lists to minimize (expected) total length
- Prefix-free coding: no code word is an initial portion of another
- Example: 11001111011



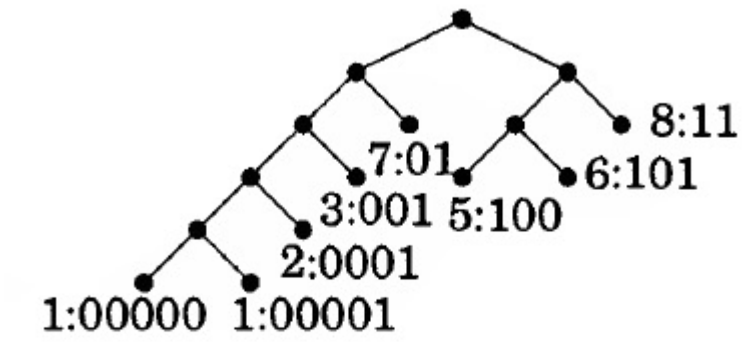
A binary prefix code for the alphabet $\{a, b, c, d, r, !\}$

Huffman's Algorithm (2.3.13, W)

- Input: Weights (frequencies or probabilities) $p_1, \ldots, p_n$
- Output: Prefix-free code (equivalently, a binary tree)
- Idea: Infrequent items should have longer codes; put infrequent items deeper by combining them into parent nodes.
- Recursion: replace the two least likely items with probabilities p, p' with a single item of weight $p + p'$

Example (2.3.14, W)

a	5	100
b	1	00000
c	1	00001
d	7	01
e	8	11
f	2	0001
g	3	001
h	6	101



The average length is $\frac{5\times 3+5+5+7\times 2+\cdots}{33} = \frac{30}{11} < 3$

Huffman coding is optimal

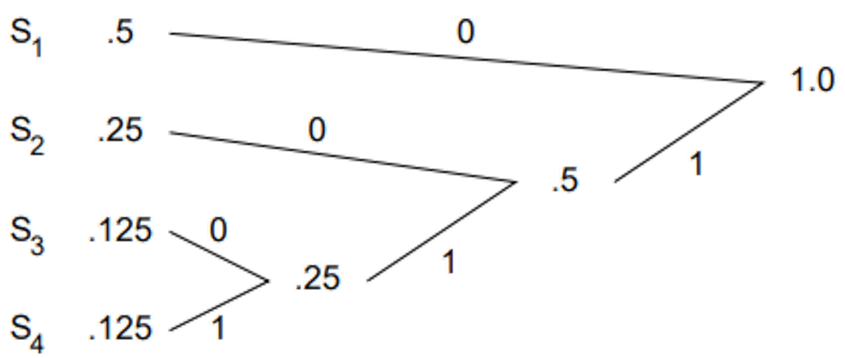
- Theorem (2.3.15, W) Given a probability distribution $\{p_i\}$ on n items, Huffman's Algorithm produces the prefix-free code with minimum expected length

Huffman coding and entropy

- The entropy of a discrete probability distribution $\{p_i\}$ is that

$$H(p) = -\sum_i p_i \log_2 p_i$$

- Exercise (Ex2.3.31, W) $H(p) \leq$ average length of Huffman coding $\leq H(p) + 1$
- Exercise (Ex2.3.30, W) When each p_i is a power of ½, average length of Huffman coding is $H(p)$



Codewords

0

10

110

111

$$\begin{aligned} \text{average length} &= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) + (3)\left(\frac{1}{8}\right) + (3)\left(\frac{1}{8}\right) \\ &= 1.75 \ \ \text{bits/symbol} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{8}\log_2 8 \\ &= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8} \\ &= 1.75 \end{aligned}$$

Lecture 4: Circuits

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Eulerian circuit

- A closed walk through a graph using every edge once is called an Eulerian circuit
- A graph that has such a walk is called an Eulerian graph

- Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree
- (possibly with multiple edges)
- Proof "$\Longrightarrow$" That G must be connected is obvious.
  Since the path enters a vertex through some edge and leaves by another edge, it is clear that all degrees must be even

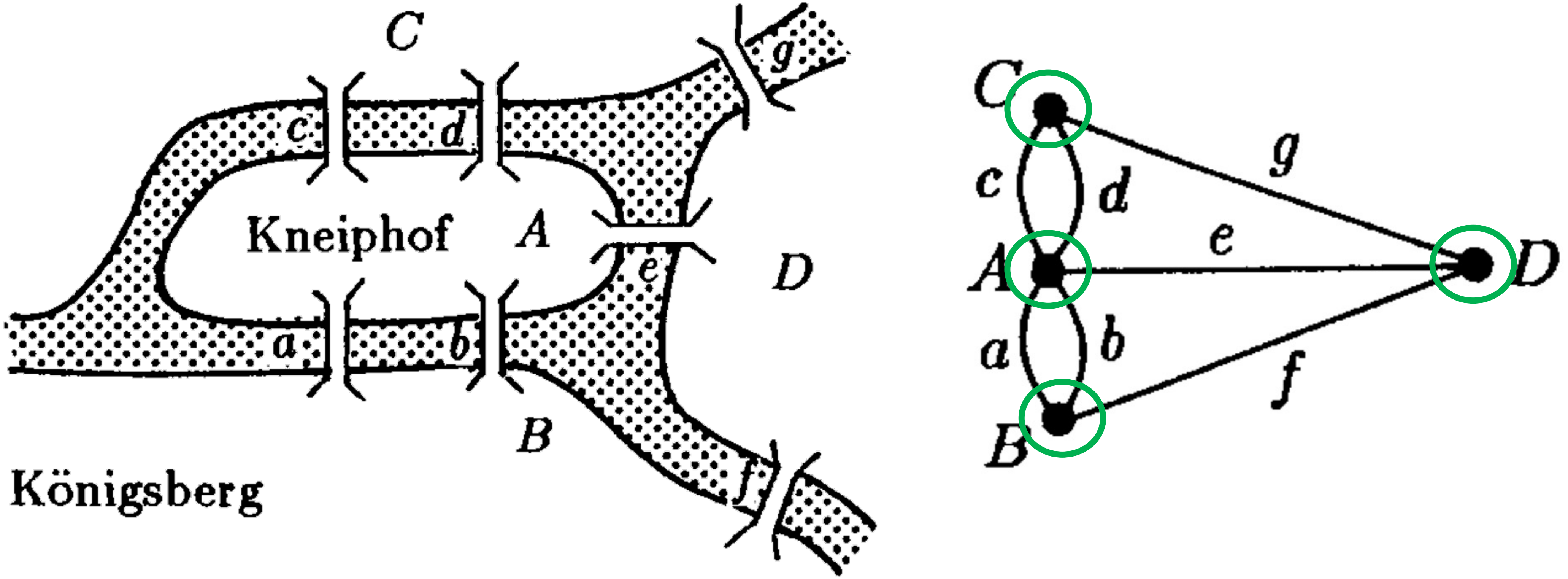
Key lemma

- Lemma (1.2.25, W) If every vertex of a graph G has degree at least 2, then G contains a cycle.

> Proposition (1.3.1, D) Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$, provided $\delta(G) \geq 2$.

Eulerian circuit

- **Theorem** (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree



Other properties

- Proposition (1.2.27, W) Every even graph decomposes into cycles
- The necessary and sufficient condition for a directed Eulerian circuit is that the graph is connected and that each vertex has the same ‘in-degree’ as ‘out-degree’

TONCAS

- TONCAS: The obvious necessary condition is also sufficient
- Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree
- Proposition (1.3.28, W) The nonnegative integers $d_1, \dots, d_n$ are the vertex degrees of some graph $\Leftrightarrow \sum_{i=1}^{n} d_i$ is even
- (Possibly with loops)
- Otherwise $(2,0,0)$ is not realizable
- **1.3.63.** (!) Let $d_1, \dots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \dots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. (Hakimi [1962])

Hamiltonian path/circuits

- A path P is Hamiltonian if $V(P) = V(G)$
  - Any graph contains a Hamiltonian path is called traceable
- A cycle C is called Hamiltonian if it spans all vertices of G
  - A graph is called Hamiltonian if it contains a Hamiltonian circuit
- In the mid-19th century, Sir William Rowan Hamilton tried to popularize the exercise of finding such a closed path in the graph of the dodecahedron (正十二面体)

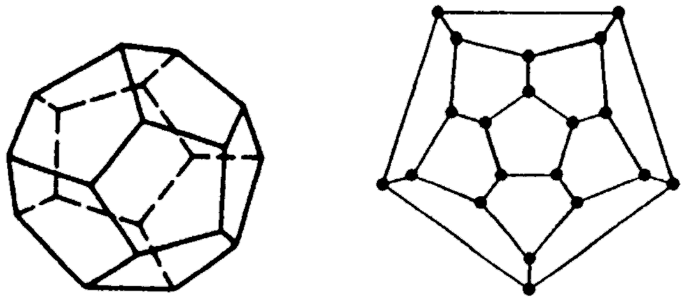
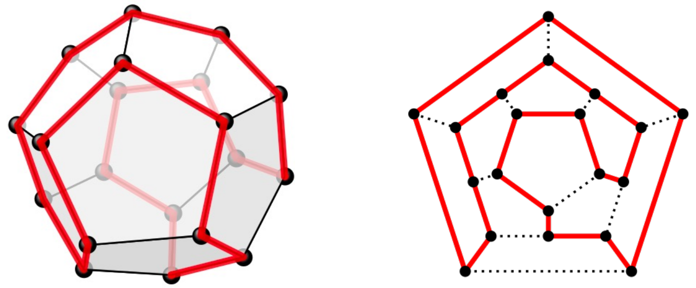


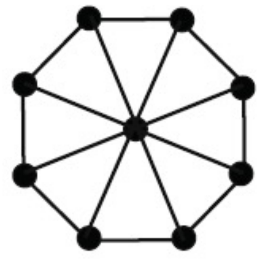
Figure 1.9



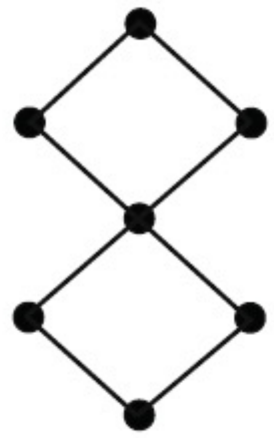
Degree parity is not a criterion

Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree

- Hamiltonian graphs
  - all even degrees C_{10}
  - all odd degrees K_{10}
  - a mixture G_1
- non-Hamiltonian graphs
  - all even G_2
  - all odd $K_{5,7}$
  - mixed P_9



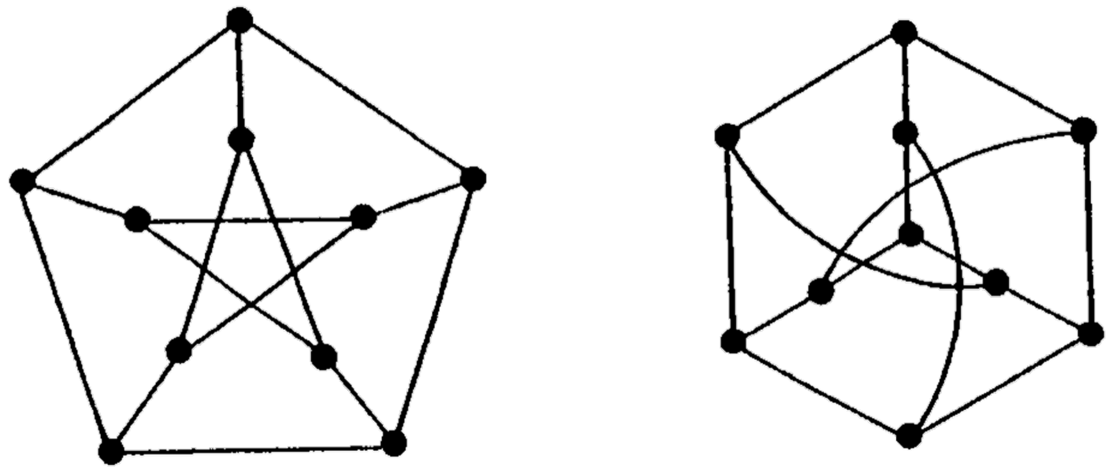
G_1



G_2

Example

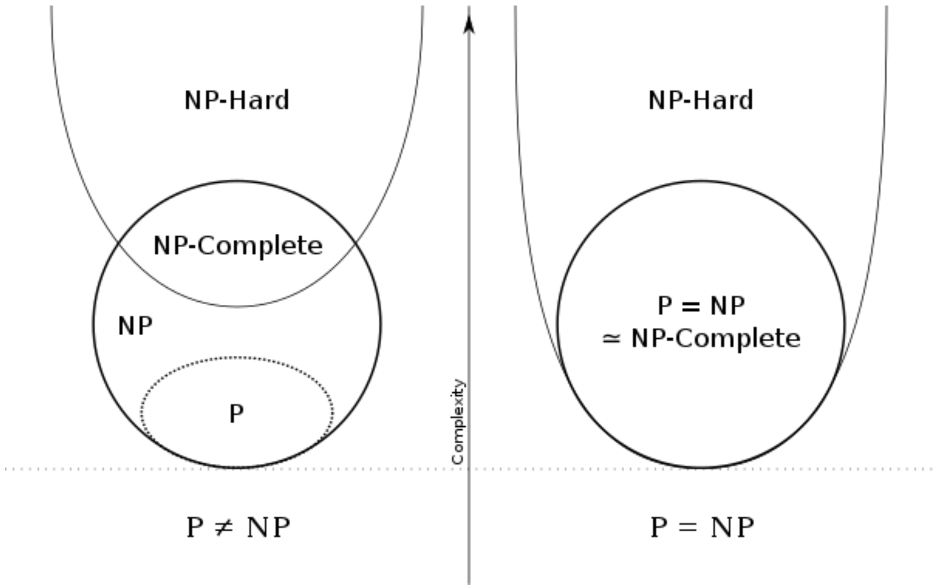
- The Petersen graph has a Hamiltonian path but no Hamiltonian cycle



- Determining whether such paths and cycles exist in graphs is the Hamiltonian path problem, which is NP-complete

P, NP, NPC, NP-hard

- P The general class of questions for which some algorithm can provide an answer in polynomial time
- NP (nondeterministic polynomial time) The class of questions for which an answer can be *verified* in polynomial time
- NP-Complete
    1. c is in NP
    2. Every problem in NP is reducible to c in polynomial time
- NP-hard
    - ~~c is in NP~~
    - Every problem in NP is reducible to c in polynomial time



Large minimal degree implies Hamiltonian

- Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected

(Ex16, S1.1.2, H) (1.3.16, W)
If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected

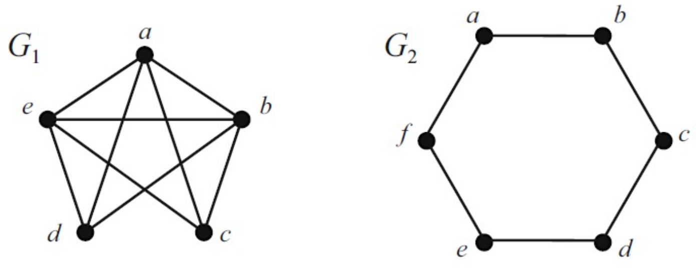
- The bound is tight
(Ex12b, S1.4.3, H) $G = K_{r,r+1}$ is not Hamiltonian
Exercise The condition when $K_{r,s}$ is Hamiltonian
- The condition is not necessary
  - C_n is Hamiltonian but with small minimum (and even maximum) degree

Generalized version

- Exercise (Theorem 1.23, H, Ore; Ex3, S1.4.3, H) Let G be a graph of order $n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all pairs of nonadjacent vertices x, y, then G is Hamiltonian

Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Independence number & Hamiltonian



- A set of vertices in a graph is called independent if they are pairwise nonadjacent
- The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set
- Example: $\alpha(G_1) = 2, \alpha(G_2) = 3$
- Theorem (1.24, H) Let G be a connected graph of order $n \geq 3$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian

(Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

Independence number & Hamiltonian 2

Theorem (1.24, H) Let G be a connected graph of order $n \geq 3$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian

- The result is tight: $\kappa(G) \geq \alpha(G)-1$ is not enough
  - $K_{r,r+1}$: $\kappa = \mathrm{r}, \alpha = r+1$
  - Exercise (Ex4, S1.4.3, H) Peterson graph: $\kappa = 3, \alpha = 4$

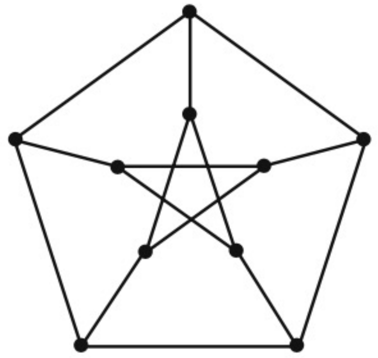
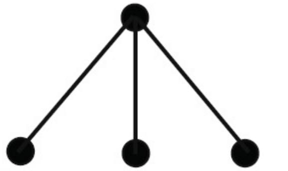
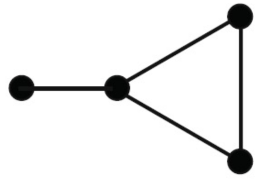


FIGURE 1.63. The Petersen Graph.

Pattern-free & Hamiltonian



$K_{1,3}$



Z_1

- G is H-free if G doesn't contain a copy of H as induced subgraph
- Theorem (1.25, H) If G is 2-connected and $\{K_{1,3}, Z_1\}$-free, then G is Hamiltonian

(Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

- The condition 2-connectivity is necessary
- (Ex2, S1.4.3, H) If G is Hamiltonian, then G is 2-connected

Lecture 5: Matchings

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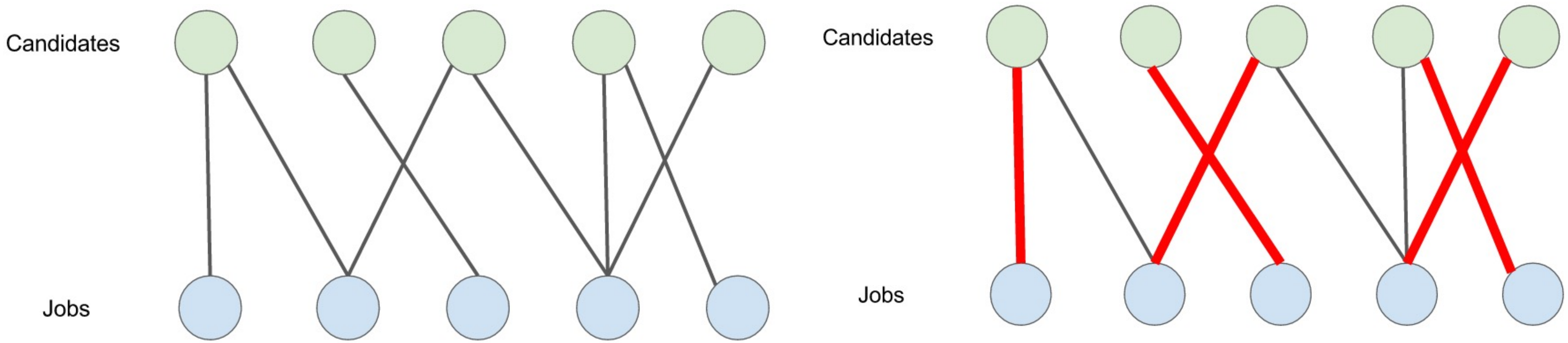
Motivating example

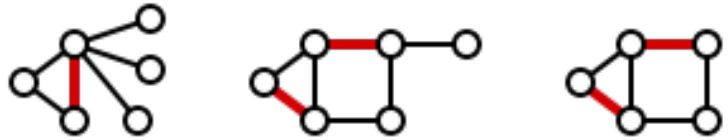


Definitions

- A matching is a set of independent edges, in which no pair of edges shares a vertex
- The vertices incident to the edges of a matching M are M-saturated (饱和的); the others are M-unsaturated
- A perfect matching in a graph is a matching that saturates every vertex
- Example (3.1.2, W) The number of perfect matchings in $K_{n,n}$ is $n!$
- Example (3.1.3, W) The number of perfect matchings in K_{2n} is
$$f_n = (2n-1)(2n-3)\cdots 1 = (2n-1)!!$$

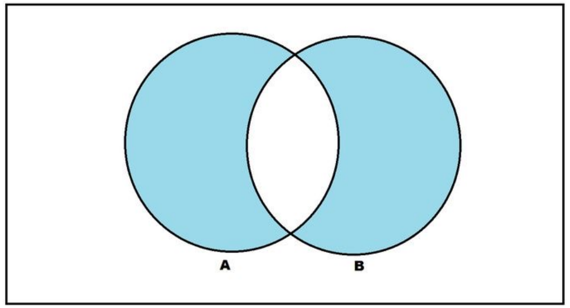
Maximal/maximum matchings 极大/最大

- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge
- A maximum matching is a matching of maximum size among all matchings in the graph
- Example: P_3, P_5

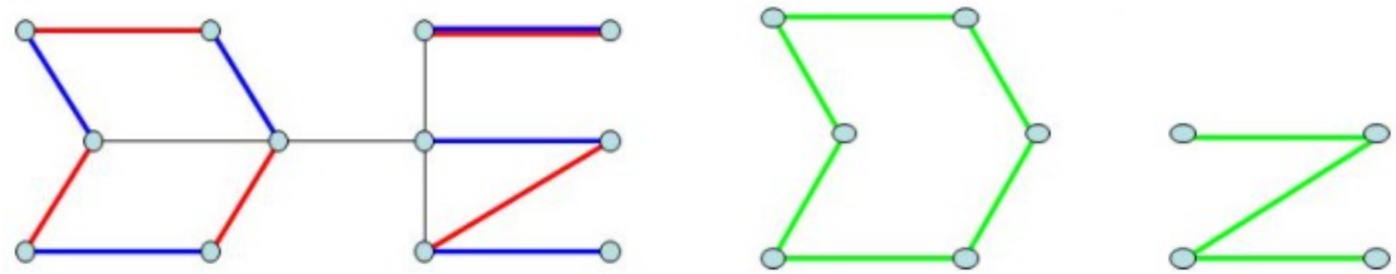


- Every maximum matching is maximal, but not every maximal matching is a maximum matching

Symmetric difference of matchings

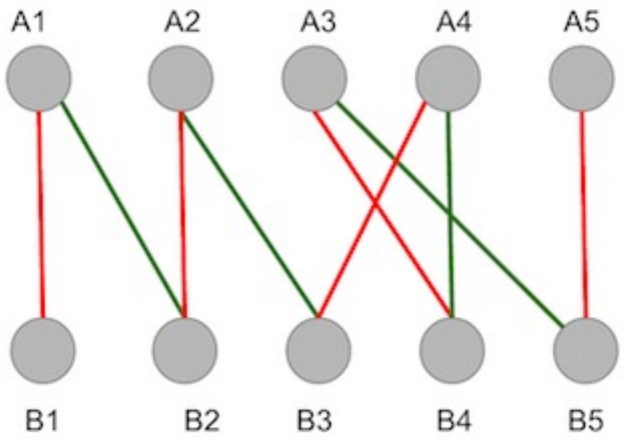


- The symmetric difference of M, M' is $M \Delta M' = (M - M') \cup (M' - M)$
- Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle

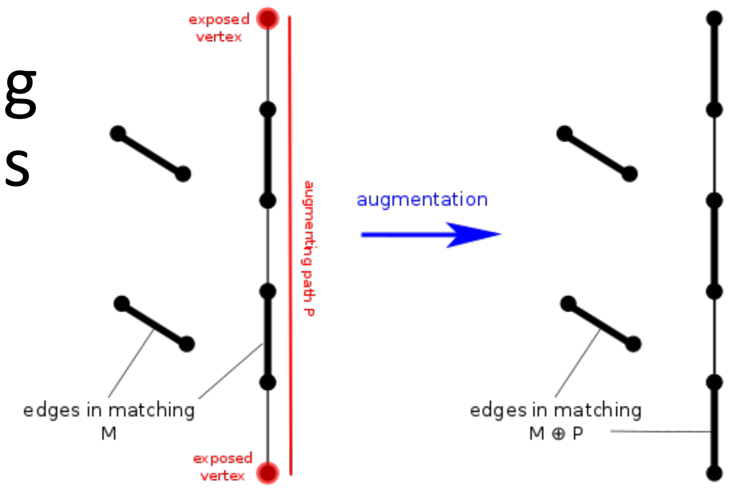


Maximum matching and augmenting path

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M
- An M-alternating path whose endpoints are M-unsaturated is an M-augmenting path
- Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path



Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



Hall's theorem (TONCAS)

- Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
  G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

- Exercise. Read the other two proofs in Diestel.
- Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

General regular graph

- Corollary (2.1.5, D) Every regular graph of positive even degree has a 2-factor
  - A k-regular spanning subgraph is called a k-factor
  - A perfect matching is a 1-factor

Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree

Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

Application to SDR

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$S_1 = \{2, 8\},$$
$$S_2 = \{8\},$$
$$S_3 = \{5, 7\},$$
$$S_4 = \{2, 4, 8\},$$
$$S_5 = \{2, 4\}.$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

> Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
> G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

König Theorem
Augmenting Path Algorithm

Vertex cover

- A set $U \subseteq V$ is a (vertex) cover of E if every edge in G is incident with a vertex in U
- Example:
  - Art museum is a graph with hallways are edges and corners are nodes
  - A security camera at the corner will guard the paintings on the hallways
  - The minimum set to place the cameras?

König-Egeváry Theorem (Min-max theorem)

- Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in G $\Leftrightarrow$ G has no M-augmenting path

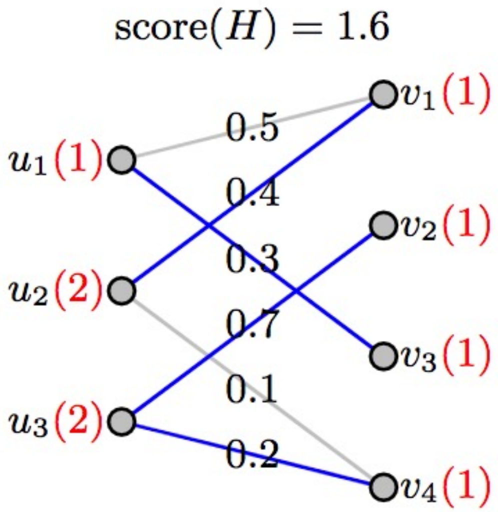
Weighted Bipartite Matching
Hungarian Algorithm

Weighted bipartite matching

- The maximum weighted matching problem is to seek a perfect matching M to maximize the total weight $w(M)$
- Bipartite graph
  - W.l.o.g. Assume the graph is $K_{n,n}$ with $w_{i,j} \geq 0$ for all $i, j \in [n]$
  - Optimization:

$$\max w(M_a) = \sum_{i,j} a_{i,j} w_{i,j}$$
$$s.t.\, a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$
$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$
$$a_{i,j} \in \{0,1\}$$

  - Integer programming
  - General IP problems are NP-Complete



(Weighted) cover

- A (weighted) cover is a choice of labels $u_1, \dots, u_n$ and $v_1, \dots, v_n$ such that $u_i + v_j \geq w_{i,j}$ for all i, j
  - The cost $c(u, v)$ of a cover (u, v) is $\sum_i u_i + \sum_j v_j$
  - The minimum weighted cover problem is that of finding a cover of minimum cost
- Optimization problem

$$\min c(u, v) = \sum_i u_i + \sum_j v_j$$
$$s.t. \; u_i + v_j \geq w_{i,j} \text{ for any } i, j$$
$$u_i, v_j \geq 0 \text{ for any } i, j$$

Duality

(IP)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\ a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$

$$a_{i,j} \in \{0,1\}$$

(Linear programming)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\ a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$

$$a_{i,j} \geq 0$$

(Dual)

$$\min \sum_{i} u_i + \sum_{j} v_j$$

$$s.t.\ u_i + v_j \geq w_{i,j} \text{ for any } i, j$$

$$u_i, v_j \geq 0$$

- Weak duality theorem
  - For each feasible solution a and (u, v)

$$\sum_{i,j} a_{i,j} w_{i,j} \leq \sum_{i} u_i + \sum_{j} v_j$$

thus $\max \sum_{i,j} a_{i,j} w_{i,j} \leq \min \sum_i u_i + \sum_j v_j$

Duality (cont.)

- Strong duality theorem
  - If one of the two problems has an optimal solution, so does the other one and that the bounds given by the weak duality theorem are tight

$$\max \sum_{i,j} a_{i,j} w_{i,j} = \min \sum_{i} u_i + \sum_{j} v_j$$

- Lemma (3.2.7, W) For a perfect matching M and cover (u, v) in a weighted bipartite graph G, $c(u, v) \geq w(M)$.
  $c(u, v) = w(M) \Leftrightarrow$ M consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  In this case, M and (u, v) are optimal.

Equality subgraph

- The equality subgraph $G_{u,v}$ for a cover (u, v) is the spanning subgraph of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  - So if $c(u, v) = w(M)$ for some perfect matching M, then M is composed of edges in $G_{u,v}$
  - And if $G_{u,v}$ contains a perfect matching M, then (u, v) and M (whose weights are $u_i + v_j$) are both optimal

Back to (unweighted) bipartite graph

- The weights are binary $0,1$
- Hungarian algorithm always maintain integer labels in the weighted cover, thus the solution will always be $0,1$
- The vertices receiving label 1 must cover the weight on the edges, thus cover all edges
- So the solution is a minimum vertex cover

Matchings in General Graphs

Perfect matchings

- K_{2n}, C_{2n}, P_{2n} have perfect matchings
- Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching
- Theorem(1.58, H) If G is a graph of order $2n$ such that $\delta(G) \geq n$, then G has a perfect matching

Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Tutte's Theorem (TONCAS)

- Let $q(G)$ be the number of connected components with odd order
- Theorem (1.59, H; 2.2.1, D; 3.3.3, W)
  Let G be a graph of order $n \geq 2$. G has a perfect matching $\Leftrightarrow q(G - S) \leq |S|$ for all $S \subseteq V$

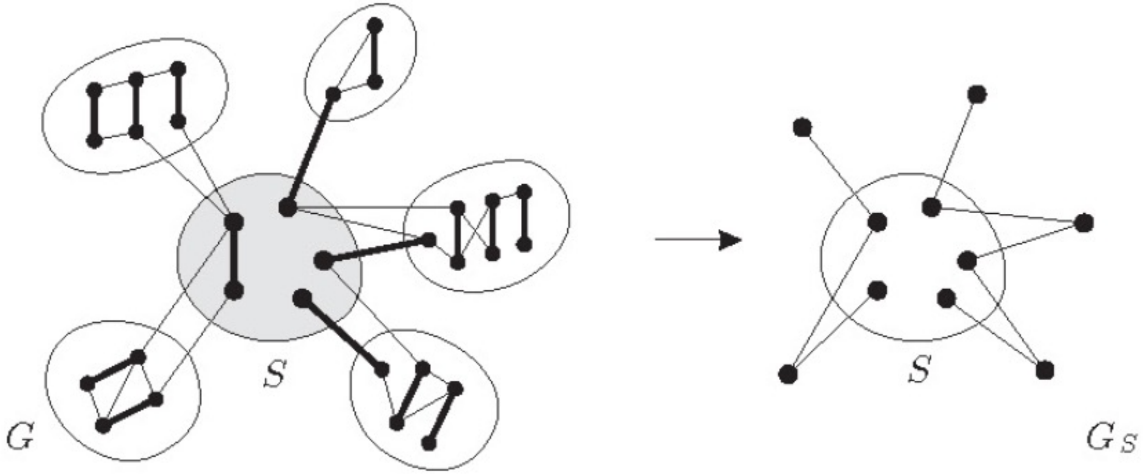


Fig. 2.2.1. Tutte's condition $q(G-S) \leqslant |S|$ for $q = 3$, and the contracted graph G_S from Theorem 2.2.3.

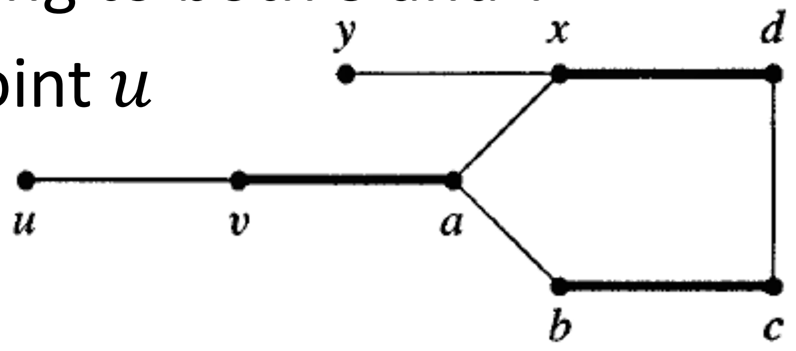
Petersen's Theorem

- Theorem (1.60, H; 2.2.2, D;3.3.8, W)
  Every bridgeless, 3-regular graph contains a perfect matching

> Theorem (1.59, H; 2.2.1, D; 3.3.3, W)
> Let G be a graph of order $n \geq 2$. G has a perfect matching $\Leftrightarrow q(G - S) \leq |S|$ for all $S \subseteq V$

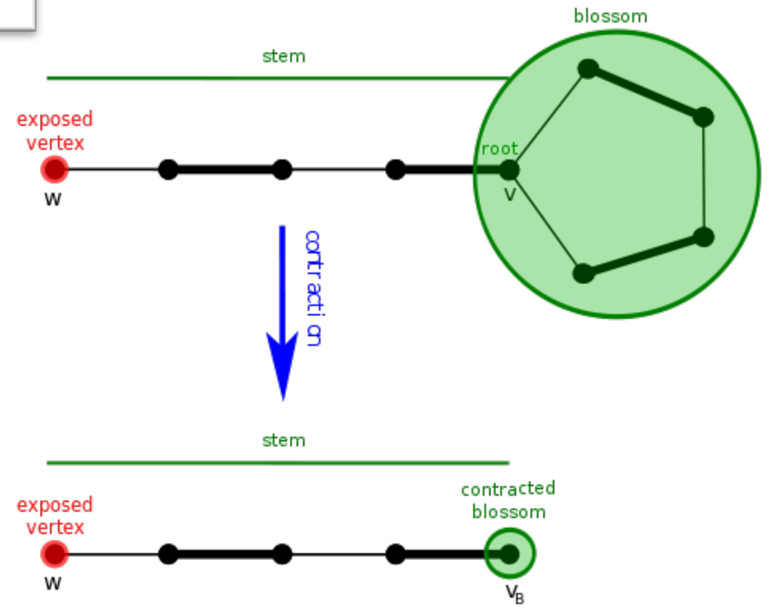
Find augmenting paths in general graphs

- Different from bipartite graphs, a vertex can belong to both S and T
- Example: How to explore from M-unsaturated point u

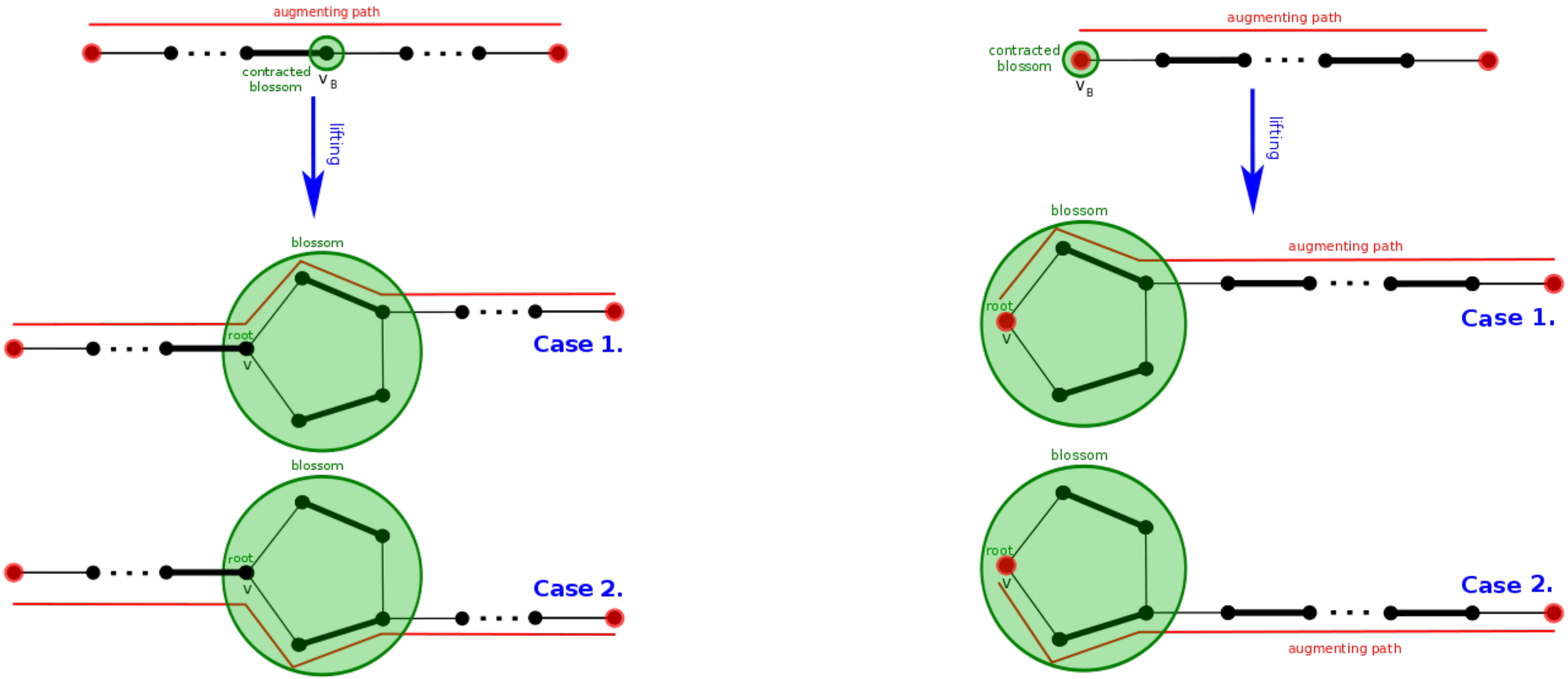


> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in G $\Leftrightarrow$ G has no M-augmenting path

- Flower/stem/blossom



Lifting


augmenting path
contracted blossom
v_B
lifting
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root
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Case 1.
blossom
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Case 2.
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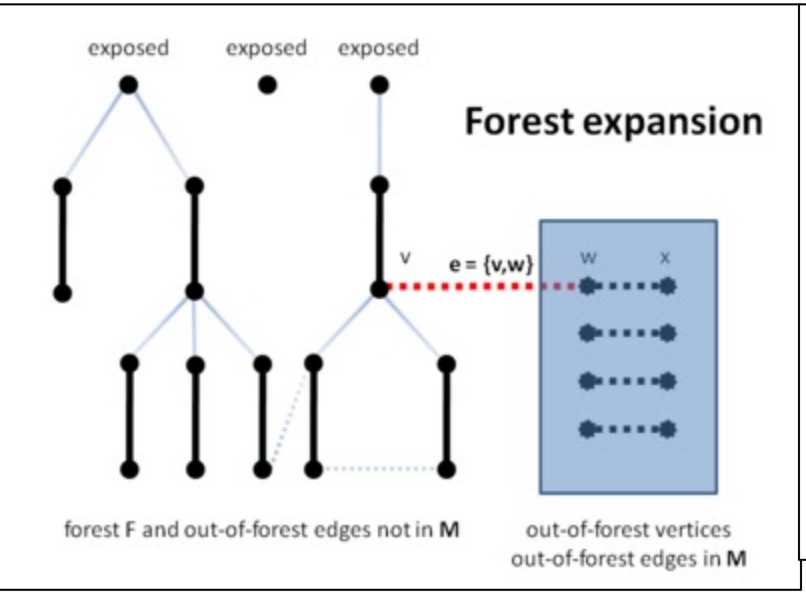
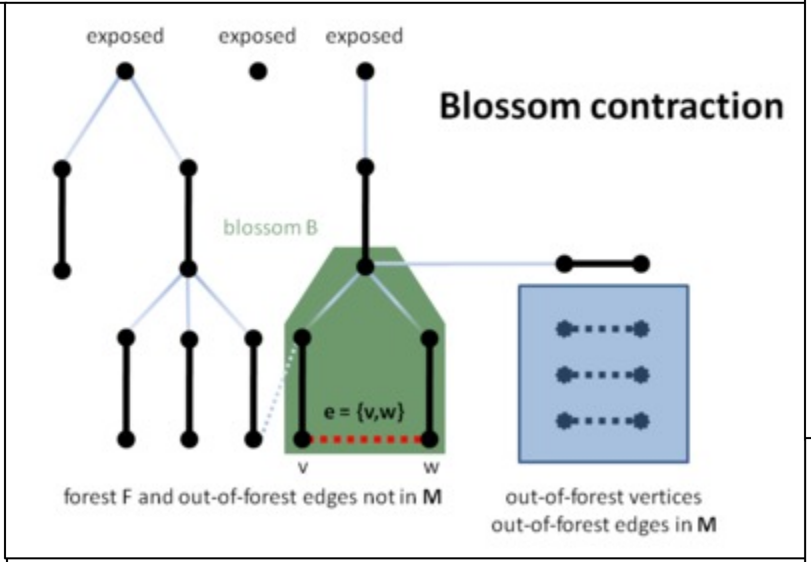
contracted blossom
v_B
lifting
blossom
augmenting path
root
v
Case 1.
blossom
root
v
Case 2.
augmenting path

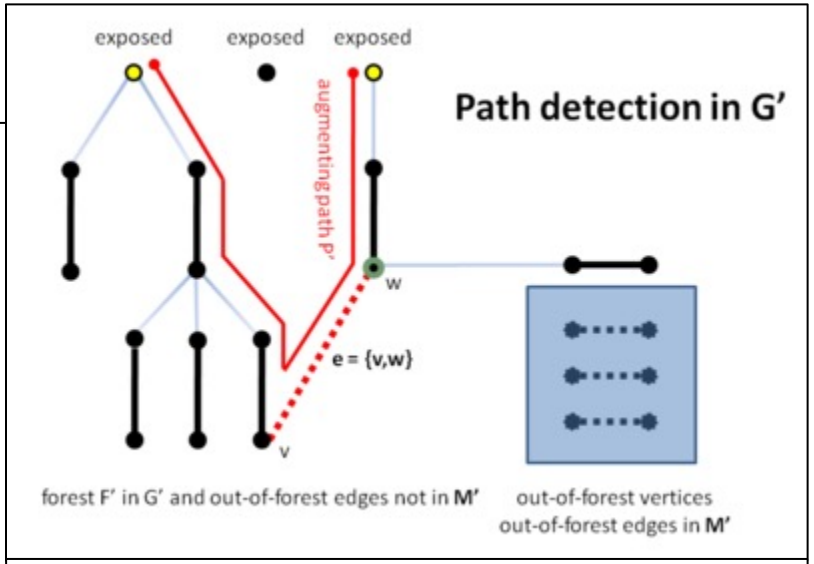
Edmonds' blossom algorithm (3.3.17, W)

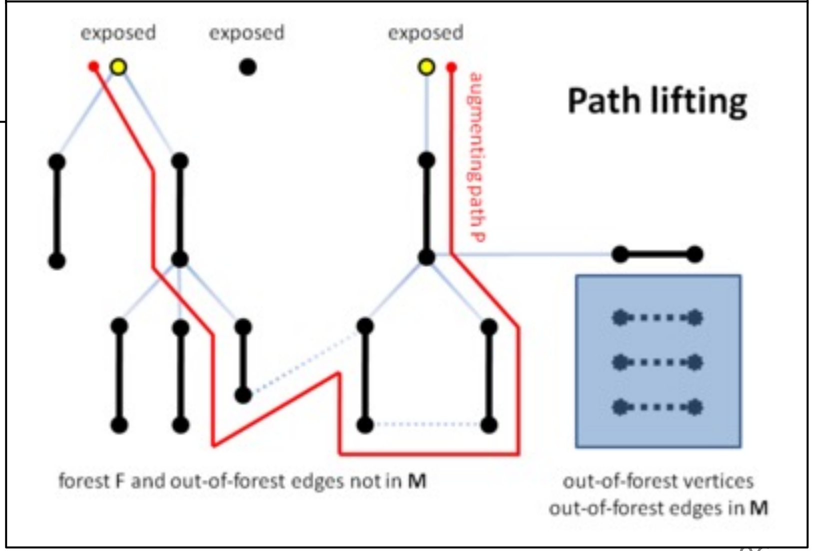
- **Input**: A graph G, a matching M in G, an M-unsaturated vertex u
- **Idea**: Explore M-alternating paths from u, recording for each vertex the vertex from which it was reached, and contracting blossoms when found
  - Maintain sets S and T analogous to those in Augmenting Path Algorithm, with S consisting of u and the vertices reached along saturated edges
  - Reaching an unsaturated vertex yields an augmentation.
- **Initialization**: $S = \{u\}$ and $T = \emptyset$
- **Iteration**: If S has no unmarked vertex, stop; there is no M-augmenting path from u
  - Otherwise, select an unmarked $v \in S$. To explore from v, successively consider each $y \in N(v)$ s.t. $y \notin T$
    - If y is unsaturated by M, then trace back from y (expanding blossoms as needed) to report an M-augmenting u, y-path
    - If $y \in S$, then a blossom has been found. Suspend the exploration of v and contract the blossom, replacing its vertices in S and T by a single new vertex in S. Continue the search from this vertex in the smaller graph.
    - Otherwise, y is matched to some w by M. Include y in T (reached from v), and include w in S (reached from y)
  - After exploring all such neighbors of v, mark v and iterate

Illustration

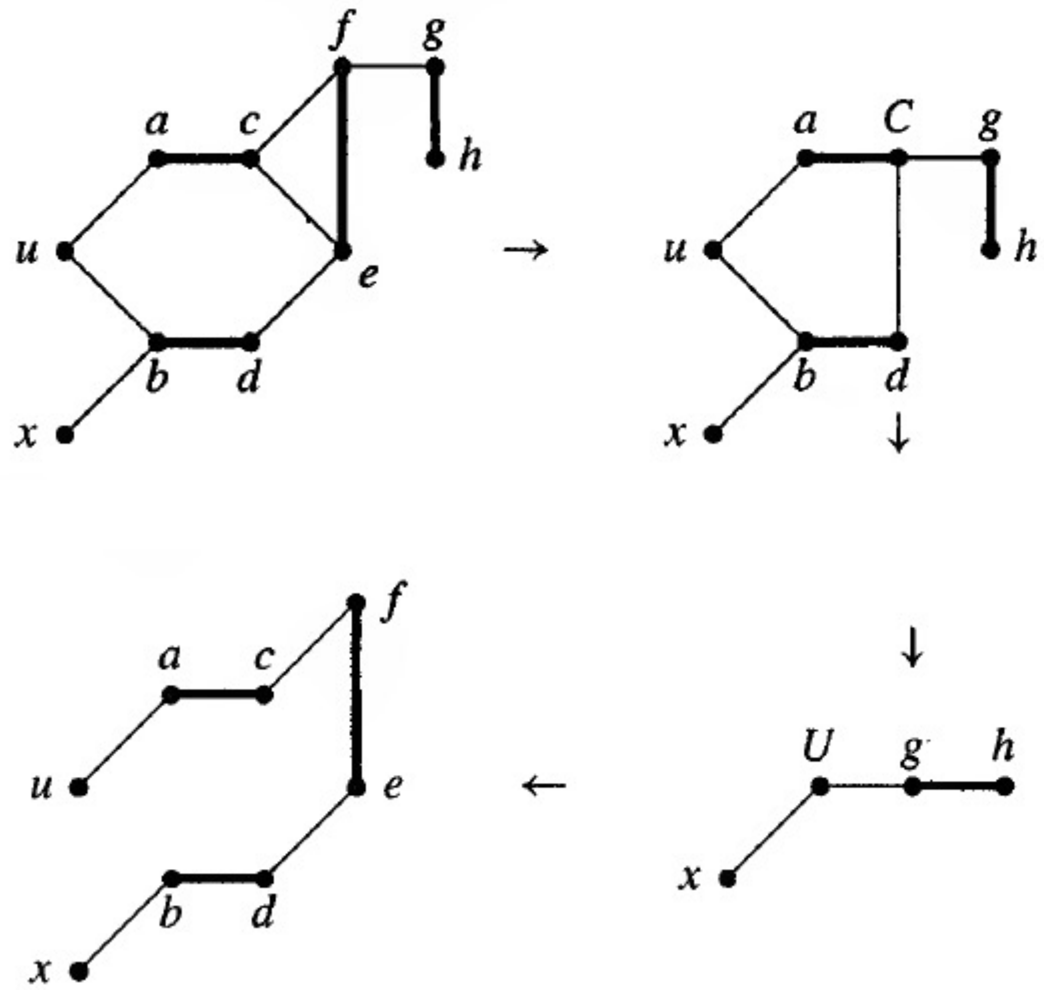




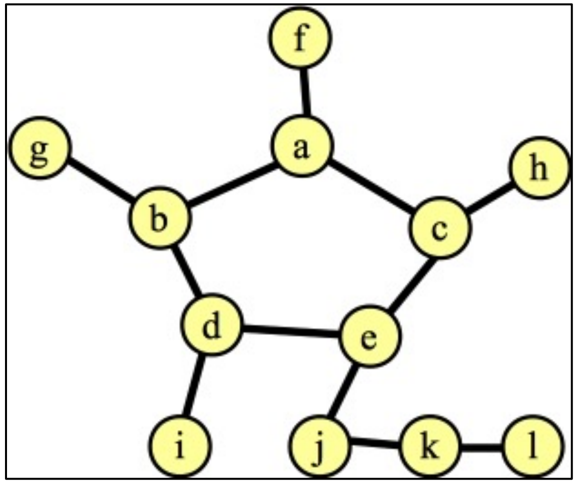




Example



Example 2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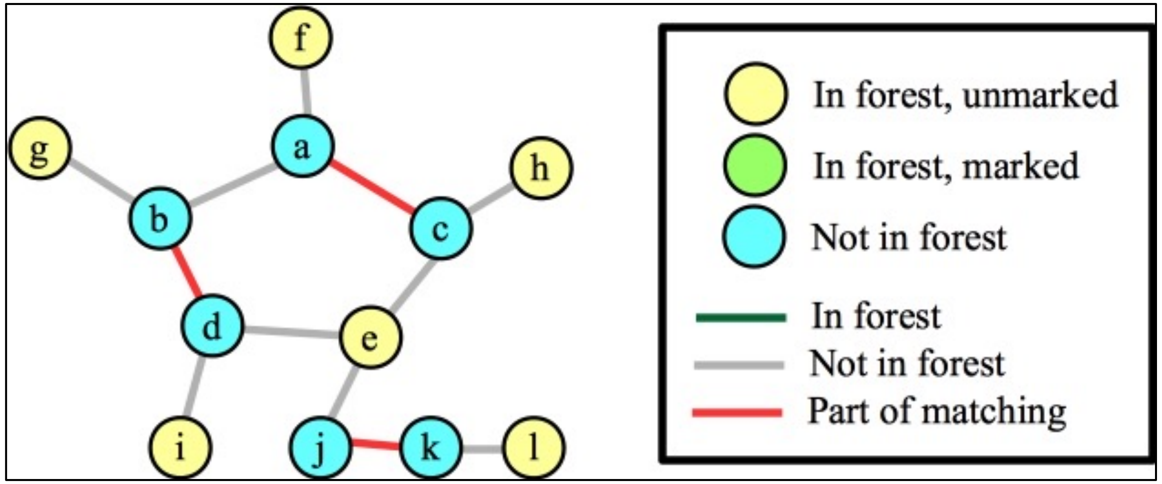
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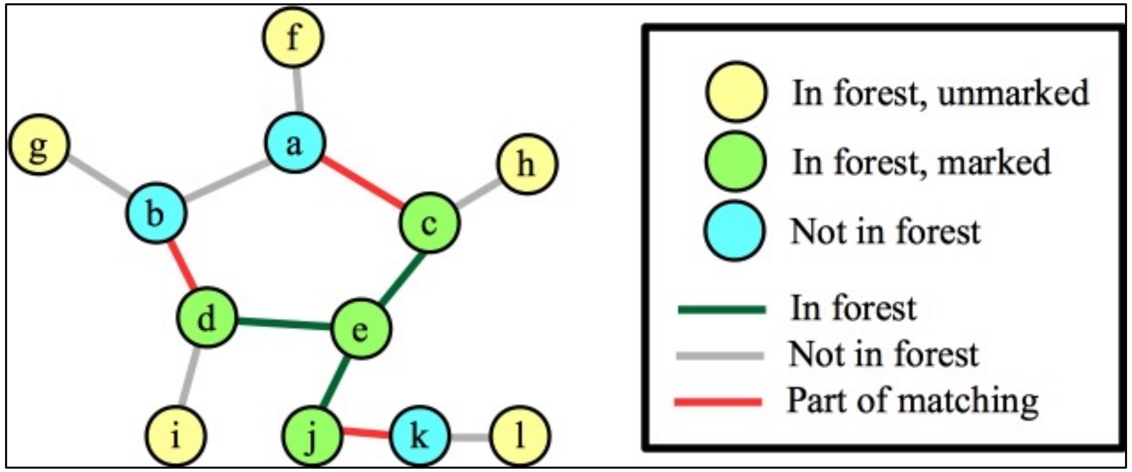
Not in forest
Part of matching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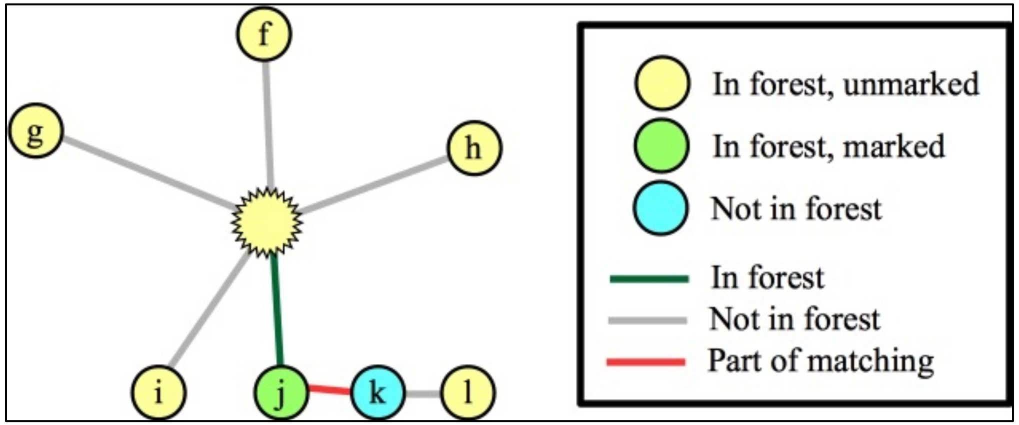
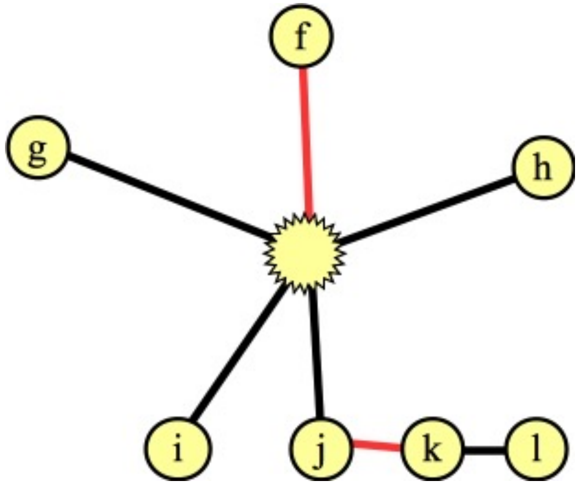
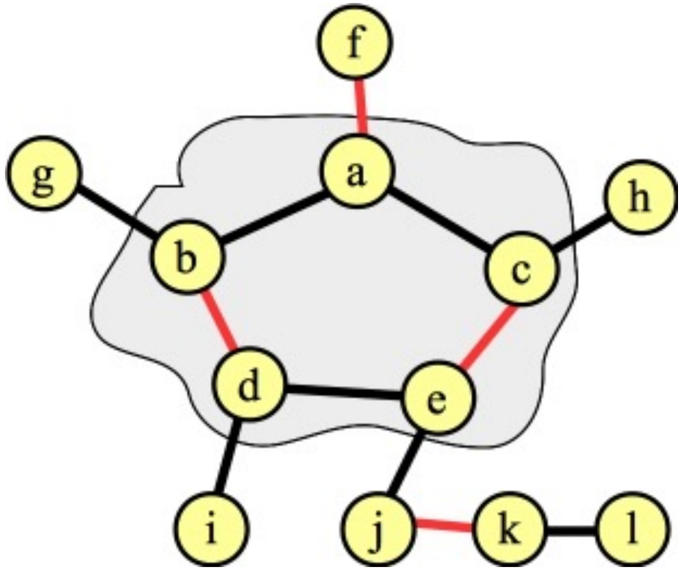
Part of matching

Example 2 (cont.)







Lecture 6: More on Connectivity

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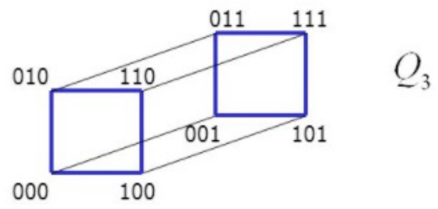
https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

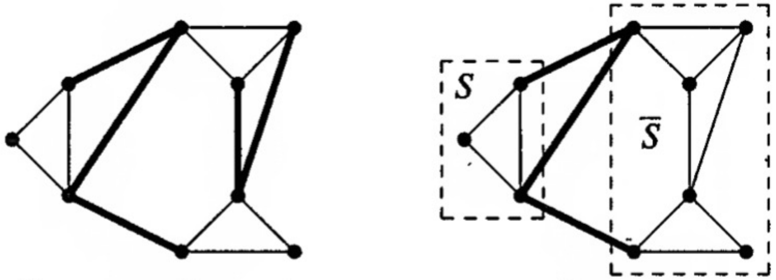
Vertex cut set and connectivity

- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected
- The connectivity, $\kappa(G)$, is the minimum size of a vertex set S of G such that $G - S$ is disconnected or has only one vertex
  - The graph is k-connected if $k \leq \kappa(G)$
- $\kappa(K_n) := n - 1$
- If G is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n, then
$$1 \leq \kappa(G) \leq n - 2$$



- For convention, $\kappa(K_1) = 0$
- Example (4.1.3, W) For k-dimensional cube $Q_k = \{0,1\}^k$, $\kappa(Q_k) = k$

Edge-connectivity



- A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component
  - A graph is k-edge-connected if every disconnecting set has at least k edges
  - The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set
- Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T
  - An edge cut is an edge set of the form $[S, S^c]$ where S is a nonempty proper subset of $V(G)$
- Every edge cut is a disconnecting set, but not vice versa
- Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Connectivity and edge-connectivity

- Proposition (1.4.2, D) If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$
- If $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$

  that is $\kappa(G) = \lambda(G) = \delta(G)$

- Theorem (4.1.11, W) If G is a 3-regular graph, then $\kappa(G) = \lambda(G)$

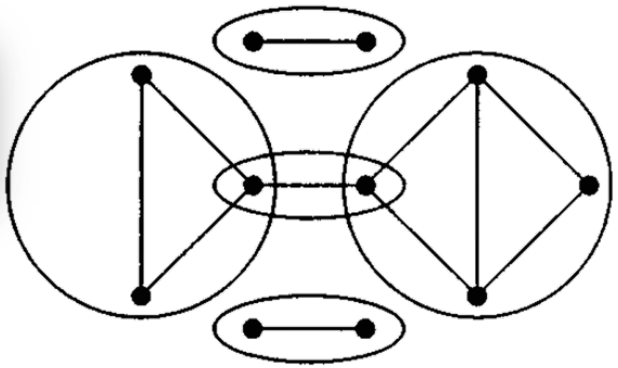
Properties of edge cut

- When $\lambda(G) < \delta(G)$, a minimum edge cut cannot isolate a vertex
- Similarly for (any) edge cut
- Proposition (4.1.12, W) If S is a set of vertices in a graph G, then
$$|[S, S^c]| = \sum_{v \in S} d(v) - 2e(G[S])$$
- Corollary (4.1.13, W) If G is a simple graph and $|[S, S^c]| < \delta(G)$, then $|S| > \delta(G)$
    - $|S|$ must be much larger than a single vertex

Blocks

- A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block
- Example

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G



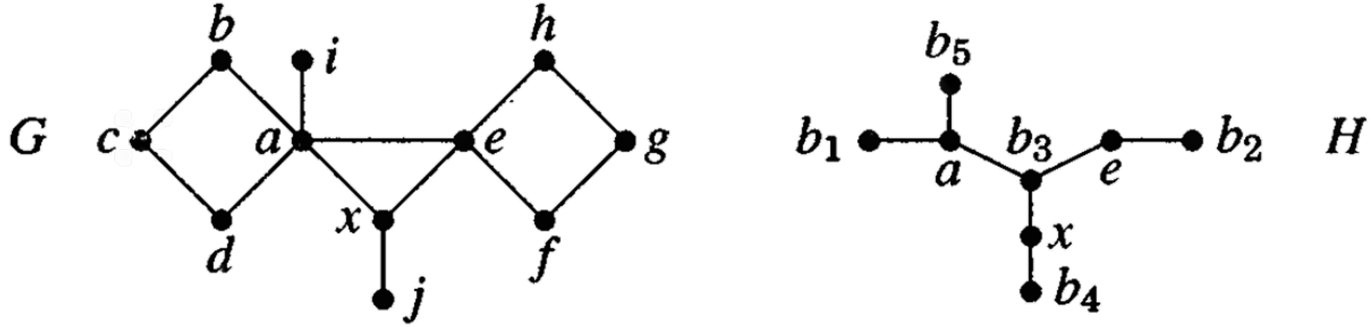
- An edge of a cycle cannot itself be a block
  - An edge is block $\Leftrightarrow$ it is a bridge
  - The blocks of a tree are its edges
- If a block has more than two vertices, then it is 2-connected
  - The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

Intersection of two blocks

- Proposition (4.1.19, W) Two blocks in a graph share at most one vertex
  - When two blocks share a vertex, it must be a cut-vertex
- Every edge is a subgraph with no cut-vertex and hence is in a block. Thus blocks in a graph decompose the edge set

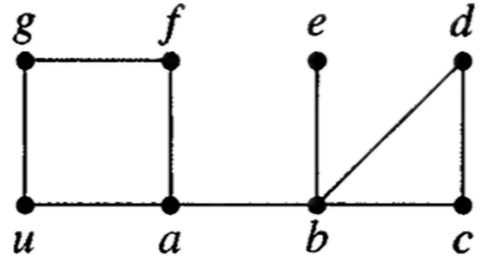
Block-cutpoint graph

- The block-cutpoint graph of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H $\Leftrightarrow$ $v \in B_i$



- (Ex34, S4.1, W) When G is connected, its block-cutpoint graph is a tree

Depth-first search (DFS)



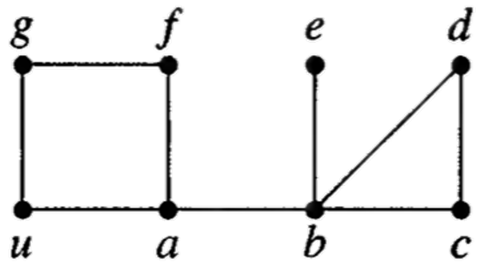
- Depth-first search

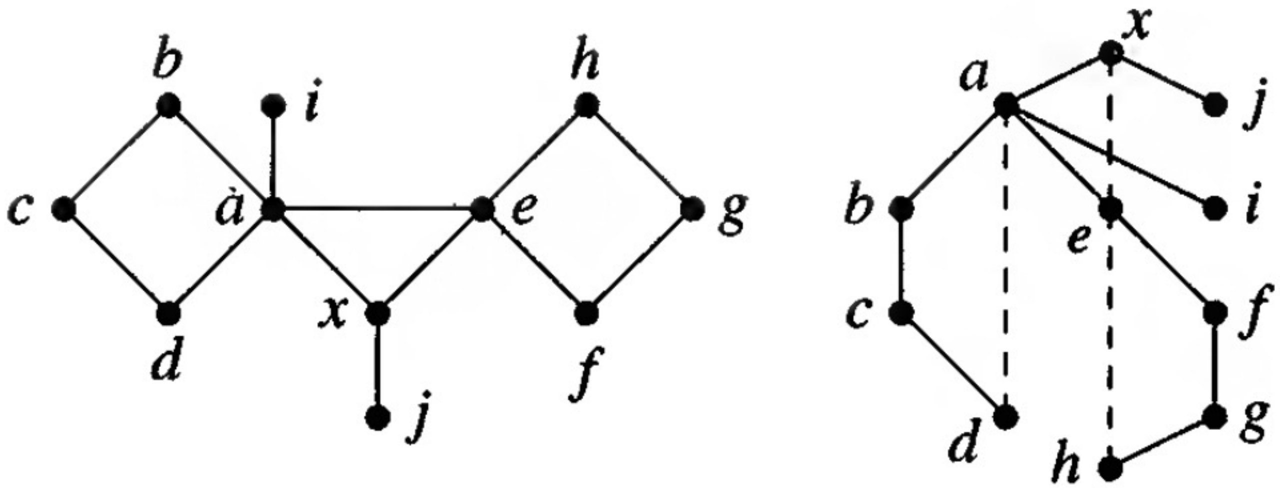
- Lemma (4.1.22, W) If T is a spanning tree of a connected graph grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T

Finding blocks by DFS

- **Input**: A connected graph G
- **Idea**: Build a DFS tree T of G, discarding portions of T as blocks are identified. Maintain one vertex called ACTIVE
- **Initialization**: Pick a root $x \in V(H)$; make x ACTIVE; set $T = \{x\}$
- **Iteration**: Let v denote the current active vertex
  - If v has an unexplored incident edge vw, then
    - If $w \notin V(T)$, then add vw to T, mark vw explored, make w ACTIVE
    - If $w \in V(T)$, then w is an ancestor of v; mark vw explored
  - If v has no more unexplored incident edges, then
    - If $v \neq x$ and w is a parent of v, make w ACTIVE. If no vertex in the current subtree T' rooted at v has an explored edge to an ancestor above w, then $V(T') \cup \{w\}$ is the vertex set of a block; record this information and delete $V(T')$
    - if $v = x$, terminate

Example





Strong orientation

- Theorem (2.5, L; 4.2.14, W; Robbins 1939) A graph has a strong orientation, i.e. an orientation that is a strongly connected digraph $\Leftrightarrow$ it is 2-edge-connected
  - A directed graph is strongly connected if for every pair of vertices (v, w), there is a directed path from v to w
  - Proposition (2.4, L) Let $xy \in T$ which is not a bridge in G and x is a parent of y. Then there exists an edge in G but not in T joining some descendant a of y and some ancestor b of x

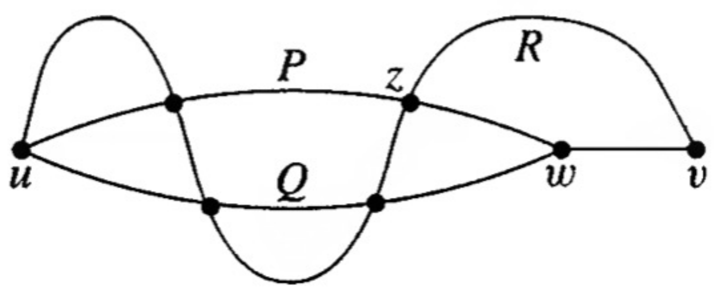
- The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

Lemma (4.1.22, W) If T is a spanning tree of a connected graph grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T

2-Connected Graphs

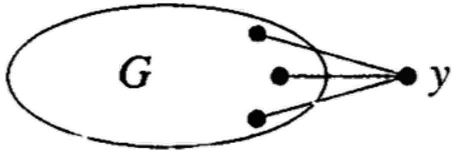
2-connected graphs

- Two paths from u to v are internally disjoint if they have no common internal vertex
- Theorem (4.2.2, W; Whitney 1932)
  A graph G having at least three vertices is 2-connected $\Leftrightarrow$ for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G

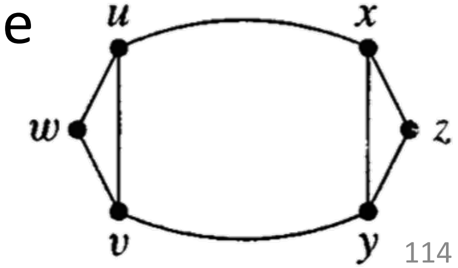


Equivalent definitions for 2-connected graphs

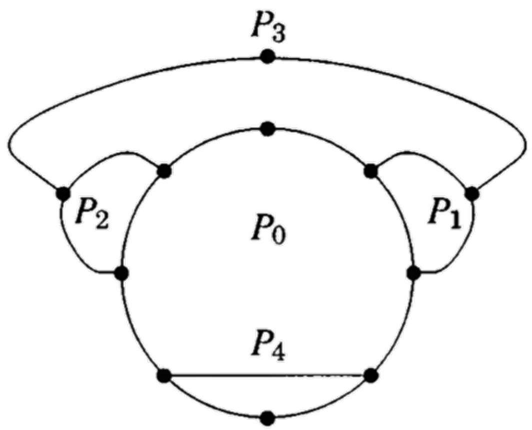
- Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



- Theorem (4.2.4, W) For a graph G with at least three vertices, TFAE
  - G is connected and has no cut-vertex
  - For all $x, y \in V(G)$, there are internally disjoint x, y-paths
  - For all $x, y \in V(G)$, there is a cycle through x and y
  - $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle



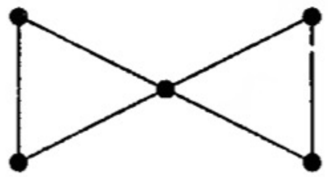
Ear decomposition



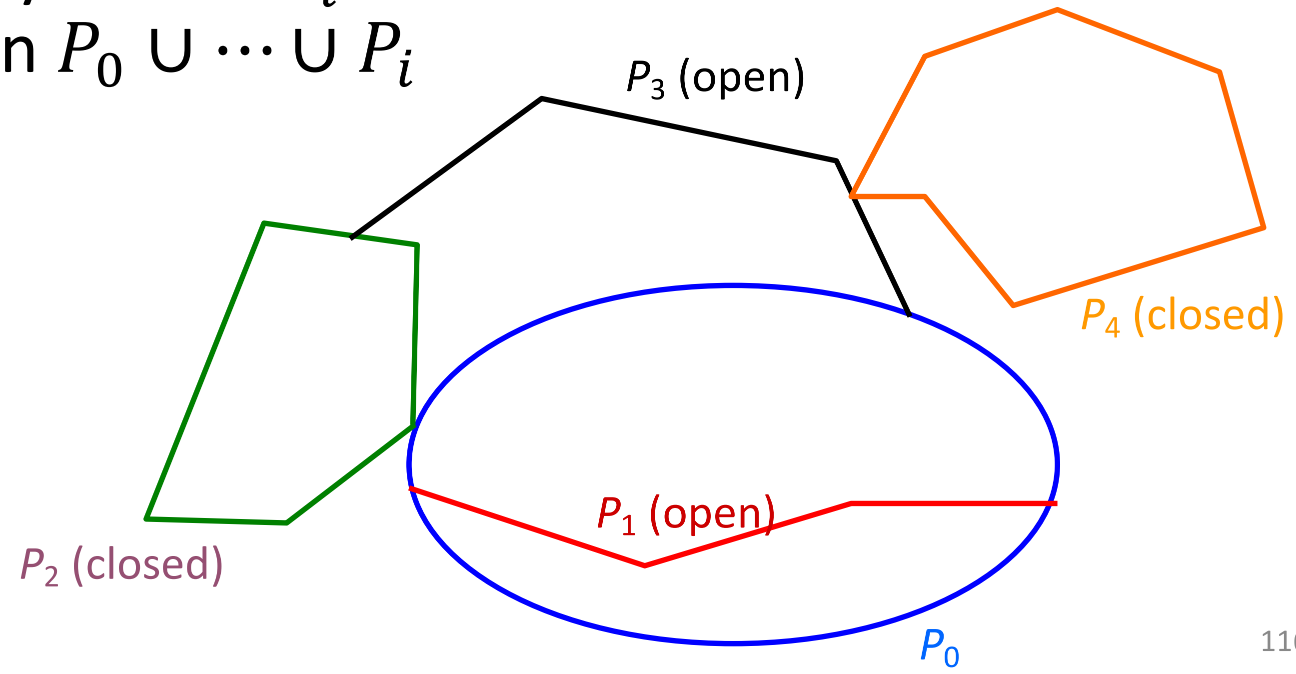
- An ear of a graph G is a maximal path whose internal vertices have degree 2 in G
- An ear decomposition of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$
- Theorem (4.2.8, W)
  A graph is 2-connected $\Leftrightarrow$ it has an ear decomposition.
  Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition
  - Corollary (4.2.6, W) If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

Closed-ear

- A closed ear of a graph G is a cycle C such that all vertices of C except one have degree 2 in G



- A closed-ear decomposition of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an (open) ear or a closed ear in $P_0 \cup \cdots \cup P_i$



Closed-ear decomposition

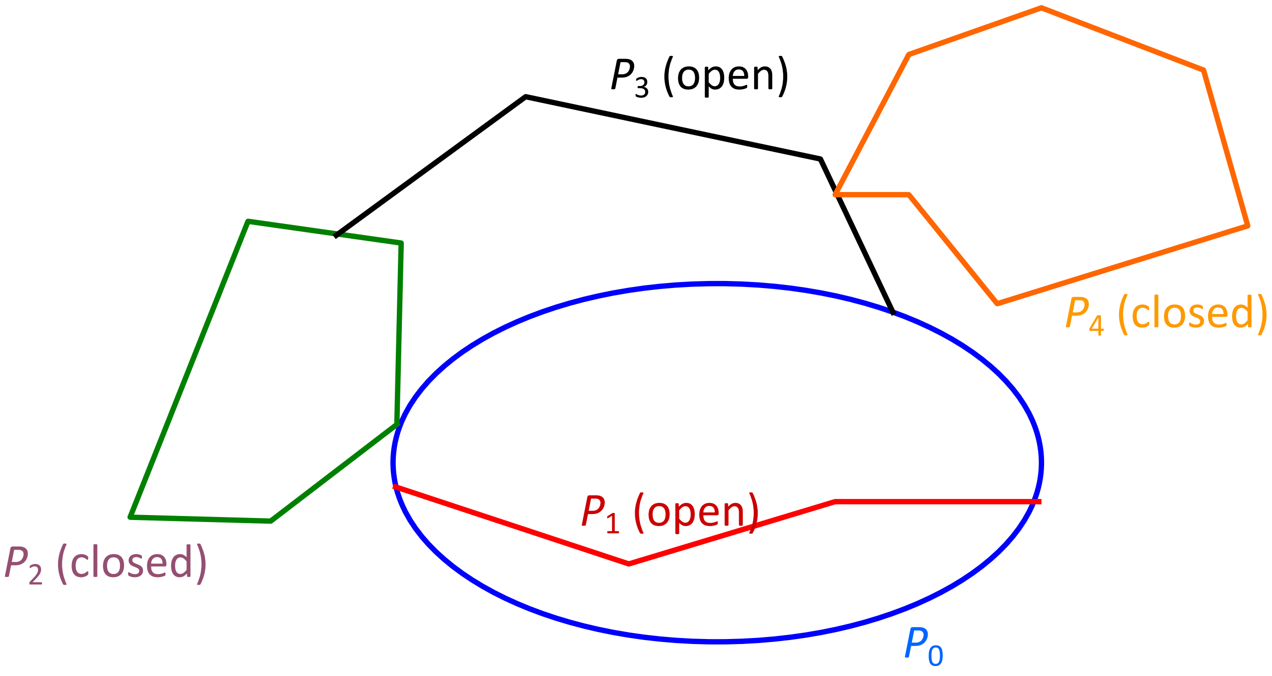
- Theorem (4.2.10, W)
  A graph is 2-edge-connected $\Leftrightarrow$ it has a closed-ear decomposition.
  Every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G

- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

Strong orientation (Revisited)

- Theorem (2.5, L; 4.2.14, W; Robbins 1939) A graph has a strong orientation, i.e. an orientation that is a strongly connected digraph $\Leftrightarrow$ it is 2-edge-connected



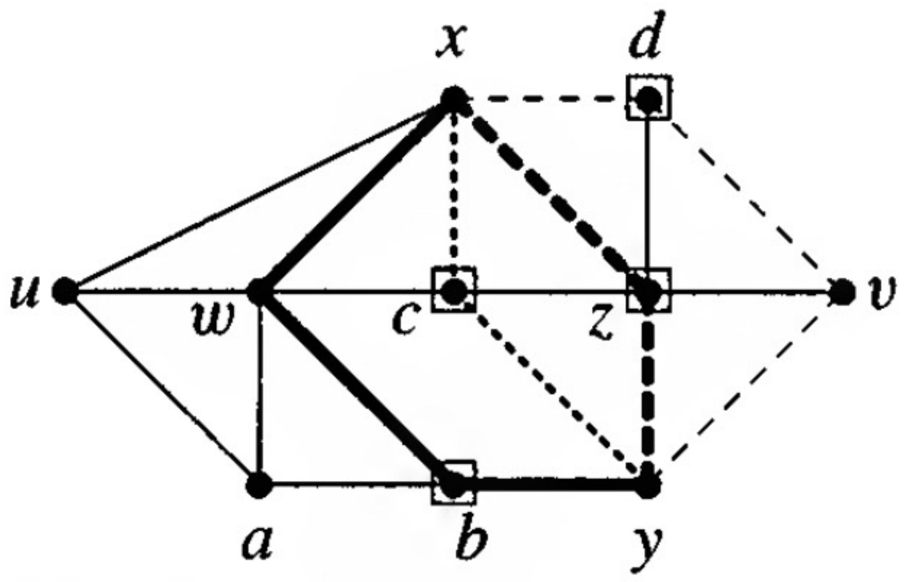
k-Connected and k-Edge-Connected graphs

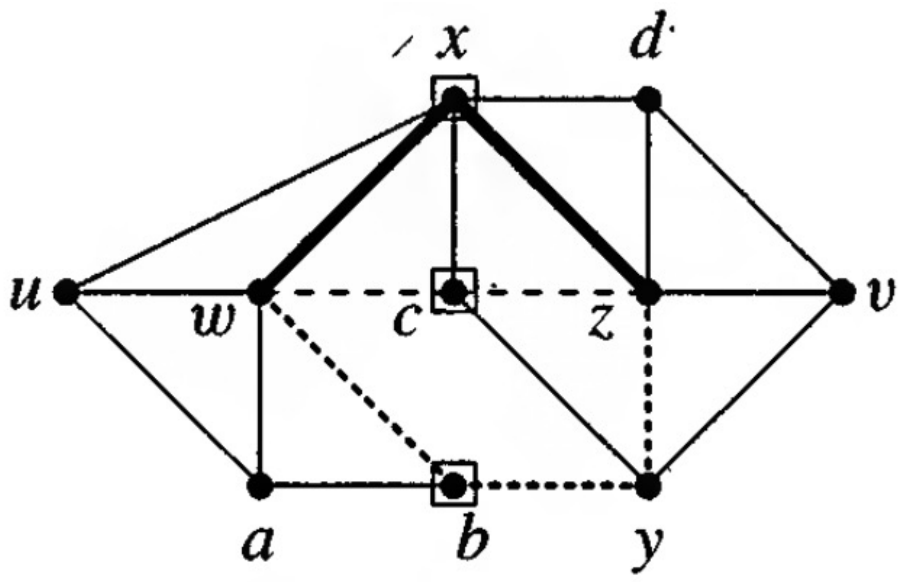
x, y-cut

- Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path
  - Let $\kappa(x, y)$ be the minimum size of an x, y-cut
  - Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths
  - $\kappa(x, y) \geq \lambda(x, y)$
- For $X, Y \subseteq V(G)$, an X, Y-path is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$

Example (4.2.16, W)

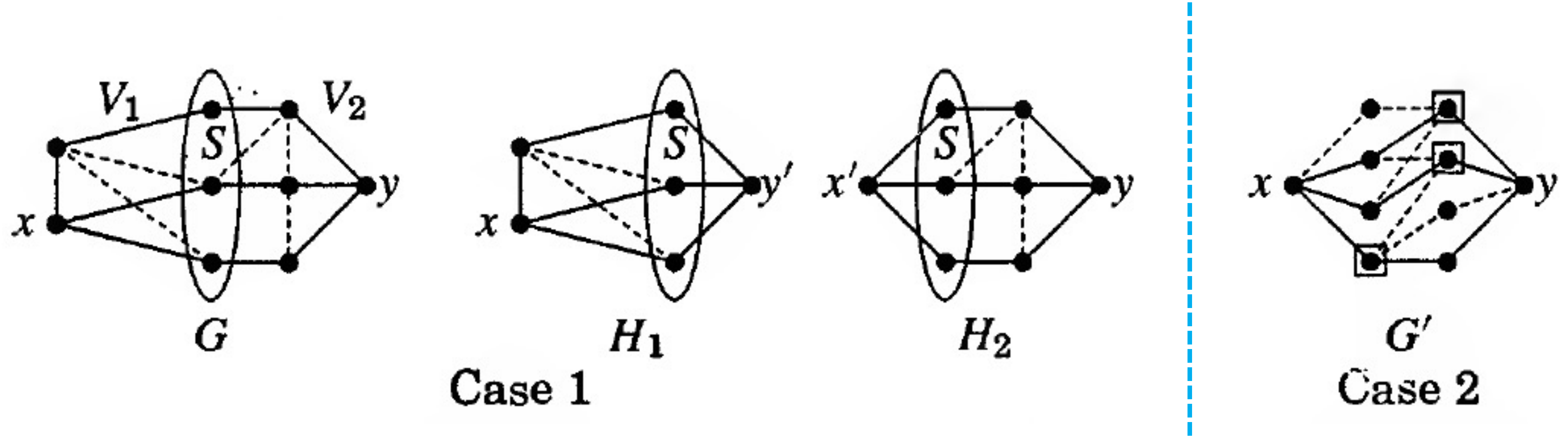
- $S = \{b, c, z, d\}$
- $\kappa(x, y) = \lambda(x, y) = 4$
- $\kappa(w, z) = \lambda(w, z) = 3$





Menger’s Theorem

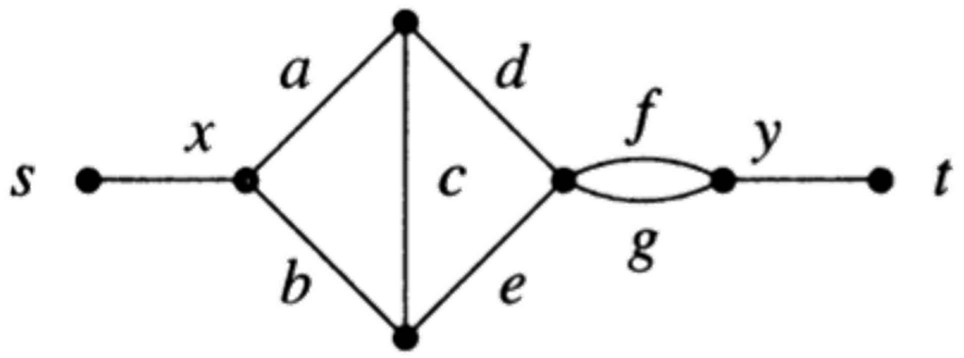
- Theorem (4.2.17, W; 3.3.1, D; Menger, 1927) If x, y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$

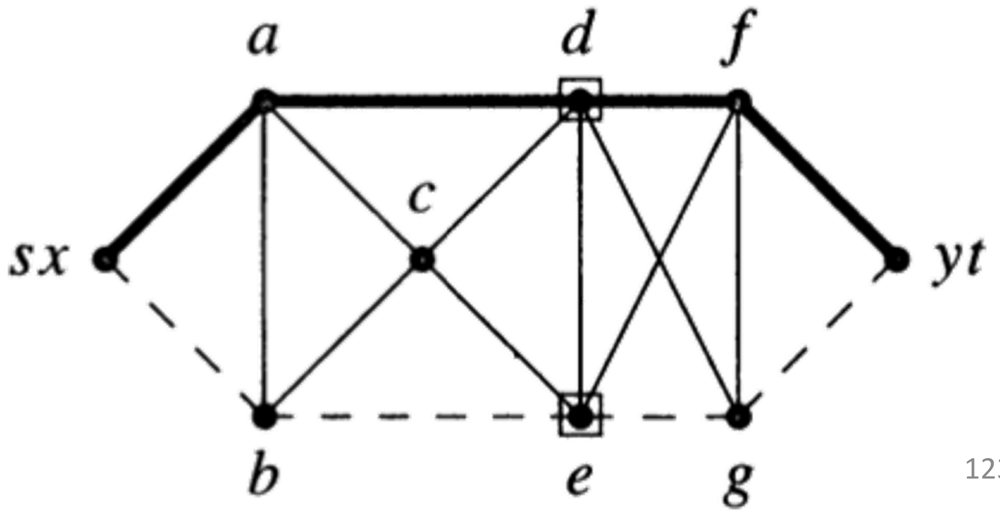


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931)
Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Edge version

- Theorem (4.2.19, W) If x and y are distinct vertices of a graph G, then the minimum size $\kappa'(x, y)$ of an x, y-disconnecting set of edges equals the maximum number $\lambda'(x, y)$ of pairwise edge-disjoint x, y-paths
  - The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G



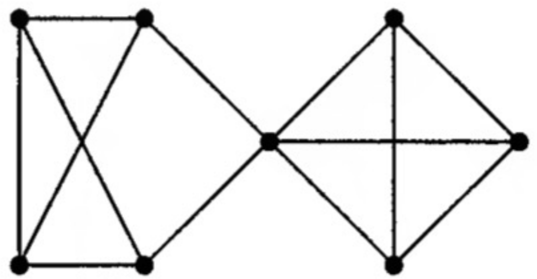


Back to connectivity

- Theorem (4.2.21, W)

$$\kappa(G) = \min_{x \neq y \in V(G)} \lambda(x, y), \qquad \lambda(G) = \min_{x \neq y \in V(G)} \lambda'(x, y)$$

  - Lemma (4.2.20, W) Deletion of an edge reduces connectivity by at most 1



Application of Menger’s Theorem

CSDR

- Let $\boldsymbol{A} = A_1, \dots, A_m$ and $\boldsymbol{B} = B_1, \dots, B_m$ be two family of sets. A common system of distinct representatives (CSDR) is a set of m elements that is both an system of distinct representatives (SDR) for $\boldsymbol{A}$ and an SDR for $\boldsymbol{B}$

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$S_1 = \{2, 8\},$$
$$S_2 = \{8\},$$
$$S_3 = \{5, 7\},$$
$$S_4 = \{2, 4, 8\},$$
$$S_5 = \{2, 4\}.$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

Equivalent condition for CSDR

- Theorem (4.2.25, W; Ford-Fulkerson 1958) Families $\boldsymbol{A} = \{A_1, \dots, A_m\}$ and $\boldsymbol{B} = \{B_1, \dots, B_m\}$ have a common system of distinct representatives (CSDR) $\Longleftrightarrow$
$$\left|\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)\right| \geq |I| + |J| - m$$
for every pair $I, J \subseteq [m]$

Lecture 7: Coloring

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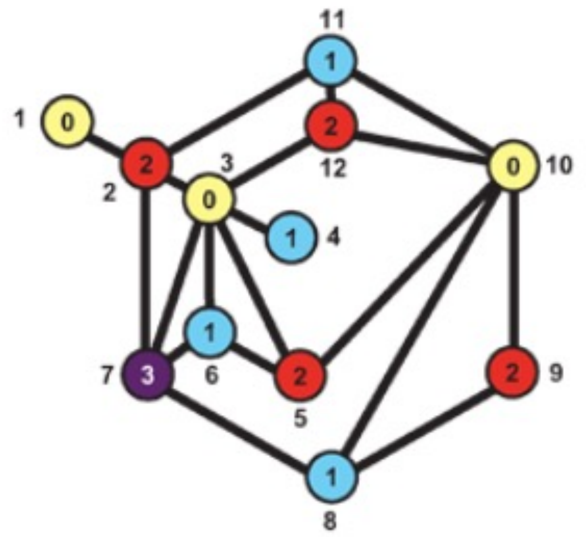
Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable
- In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. G is said to be k-chromatic
- Examples

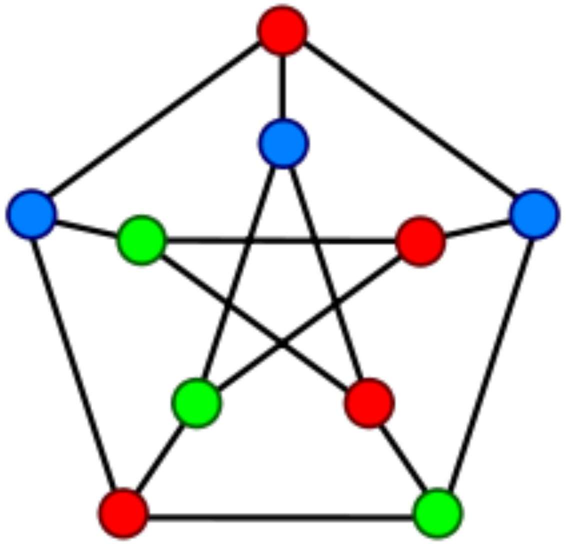
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$\chi(K_n) = n,$

$\chi(E_n) = 1,$ ←Empty graph

$\chi(K_{m,n}) = 2.$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

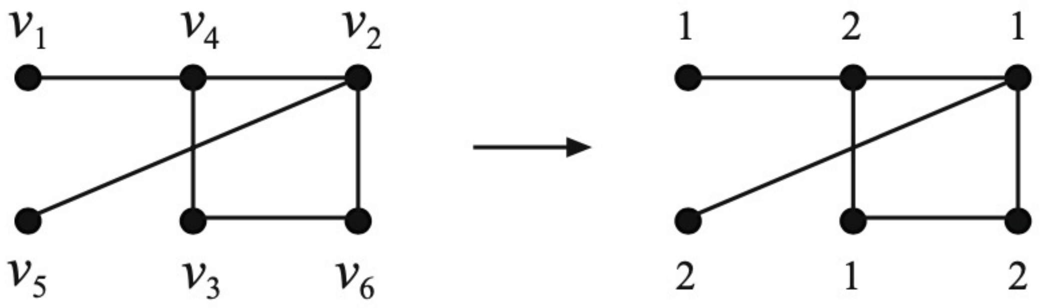
Bounds on Chromatic number

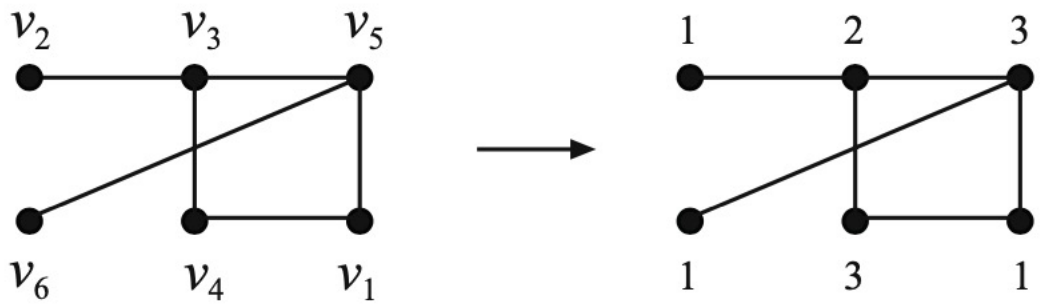
- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \ldots, v_n$
- Next, order the available colors $(1,2,\ldots,n)$ in some way
  - Start coloring by assigning color 1 to vertex v_1
  - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
  - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors



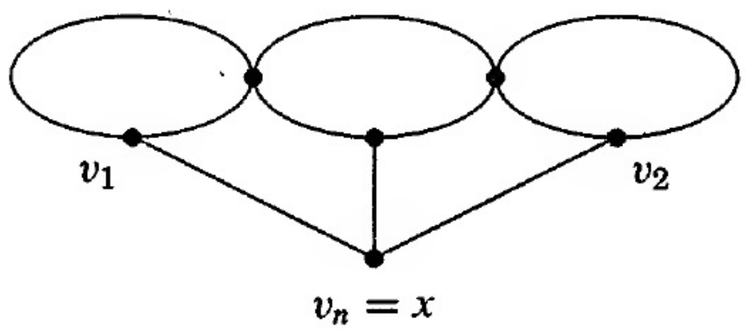


Bound using the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
  The equality is obtained for complete graphs and odd cycles

Brooks's theorem

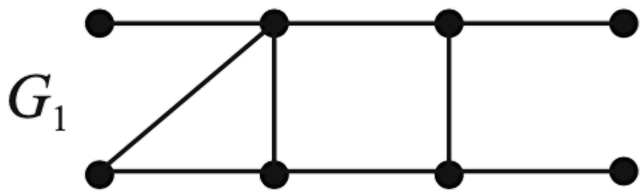
- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$

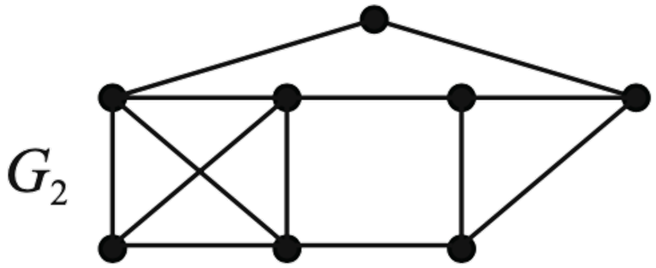


- ⇒The Petersen graph is 3-colorable

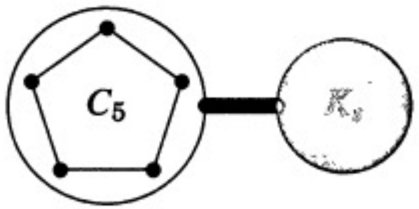
Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$





- Theorem (1.44, H; 5.1.7, W) For any graph G, $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



Chromatic number and independence number

- Theorem (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \le \chi(G) \le n + 1 - \alpha(G)$$

The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

Extremal properties for k-chromatic graphs

- Proposition (5.2.5, W) Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges
  - Equality holds for a complete graph plus isolated vertices.

> In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

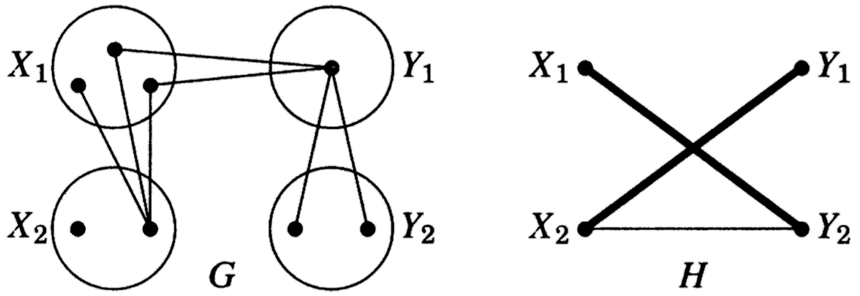
- The Turán graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ by at most 1 vertex
  - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- Lemma (5.2.8, W) Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges
- Turán's Theorem (5.2.9, W; Turán 1941) Among the n-vertex simple K_{r+1}-free graphs, $T_{n,r}$ has the maximum number of edges

Color-critical

- If $\chi(H) < \chi(G) = k$ for every proper subgraph H, then G is color-critical or k-critical
- K_2 is the only 2-critical graph
  K_1 is the only 1-critical graph
- (5.2.12, W) A graph with no isolated vertices is color-critical $\Leftrightarrow$
  $\chi(G-e) < \chi(G)$ for every edge $e \in E(G)$
- Proposition (5.2.13, W) Let G be a k-critical graph
  (a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k-1$ colors all appear on $N(v)$
  $\Rightarrow \delta(G) \geq k-1$
  (b) For every $e \in E(G)$, every proper $(k-1)$-coloring of $G-e$ gives the same color to the two endpoints of e

Color-critical has edge-connectivity

- Theorem (5.2.16, W; Dirac 1953) Every k-critical graph is $(k-1)$-edge-connected
- Lemma (5.2.15, W; Kainen) Let G be a graph with $\chi(G) > k$ and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges

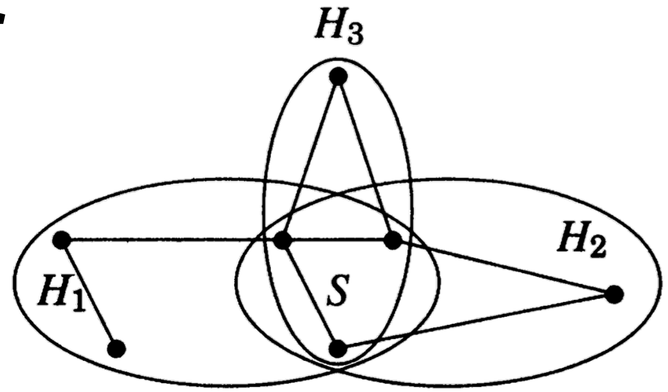


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Color-critical and vertex cut set

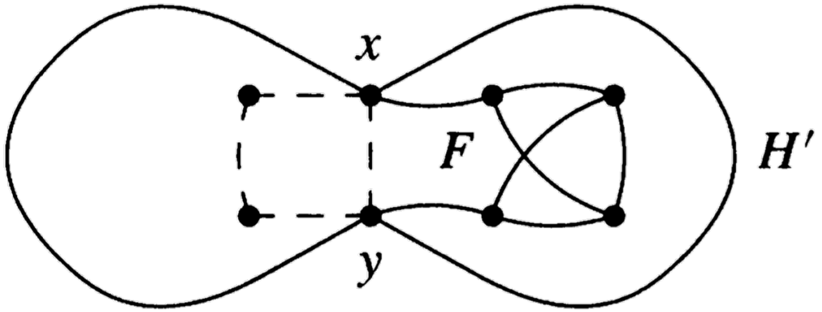
- Let S be a set of vertices in a graph G. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component in $G - S$

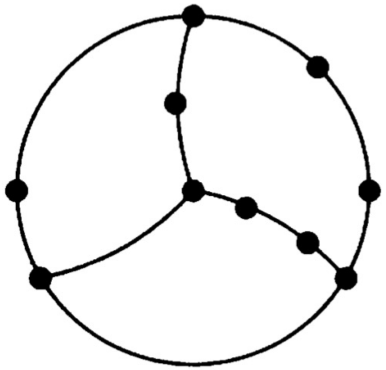


- Proposition (5.2.18, W) If G is k-critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S-lobe H such that $\chi(H + xy) = k$

Chromatic number 4 has a K_4-subdivision

- Theorem (5.2.20, W; Dirac 1952) Every graph with chromatic number at least 4 contains a K_4-subdivision

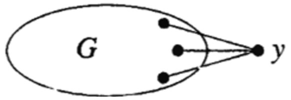




a subdivision of K_4

Proposition (5.2.18, W) If G is k-critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S-lobe H such that $\chi(H + xy) = k$

Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



Hajós' conjecture

- Hajós' conjecture [1961]: Every k-chromatic graph contains a subdivision of K_k
- $k = 2$: Every 2-chromatic graph has a nontrivial path
- $k = 3$: Every 3-chromatic graph has a cycle
- It is open for $k = 5,6$
- Exercise (Ex5.2.40, W) It is false for $k = 7$ or 8

Chromatic Polynomials

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $\chi(G; k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
  - How many different colorings of K_4 using 4 colors?
    - $4\times3\times2\times1$
    - $\chi(K_4; 4) = 24$
  - How many different colorings of K_4 using 6 colors?
    - $6\times5\times4\times3$
    - $\chi(K_4; 6) = 360$
  - How many different colorings of K_4 using 2 colors?
    - 0
    - $\chi(K_4; 2) = 0$

Examples

- If $k \geq n$
$$\chi(K_n; k) = k(k-1)\cdots(k-n+1)$$
- If $k < n$
$$\chi(K_n; k) = 0$$
- G is k-colorable $\Leftrightarrow$ $\chi(G) \leq k$ $\Leftrightarrow \chi(G; k) > 0$
- $\chi(G) = \min\{k \geq 1: \chi(G; k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

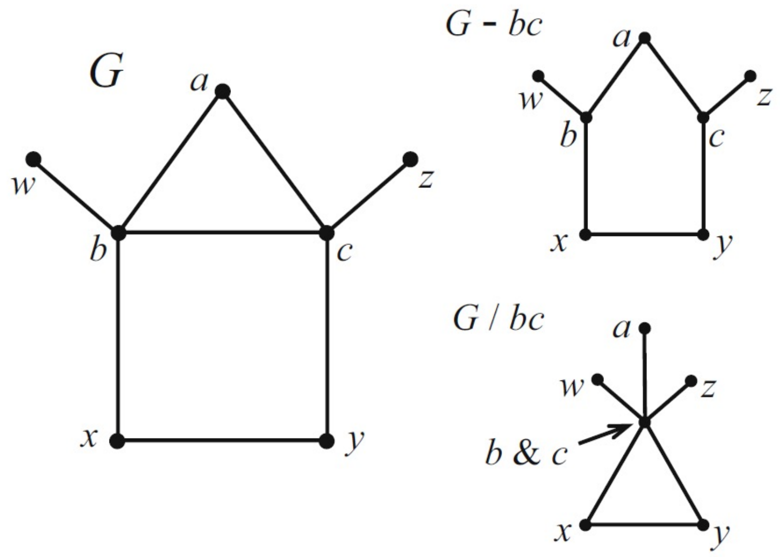


FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G. Then

$$\chi(G;k) = \chi(G-e;k) - \chi(G/e;k)$$

Use chromatic recurrence to compute $\chi(G; k)$

- Example: Compute $\chi(P_3; k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $\chi(P_3; 1) = 0, \chi(P_3; 2) = 2$

FIGURE 1.102. Two 2-colorings of P_3

- Example: What is $\chi(K_n - e; k)$?

More examples

- Path P_{n-1} has $n-1$ edges (n vertices)
$$\chi(P_{n-1}; k) = k(k-1)^{n-1}$$
- Any tree T on n vertices
$$\chi(T; k) = k(k-1)^{n-1}$$
- Cycle C_n
$$\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$$
  - When n is odd, $\chi(C_n; 2) = 0, \chi(C_n; 3) > 0$
  - When n is even, $\chi(C_n; 2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $\chi(G; k)$ is a polynomial in k of degree n
  - The leading coefficient of $\chi(G; k)$ is 1
  - The constant term of $\chi(G; k)$ is 0
    - If G has i components, then the coefficients of $k^0, \dots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $\chi(G; k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow$ $\chi(G; k) = k(k-1)^{n-1}$

      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n - 1$ edges

    - A graph G is complete $\Leftrightarrow$ $\chi(G; k) = k(k-1)\cdots(k-n+1)$

Simplicial elimination ordering

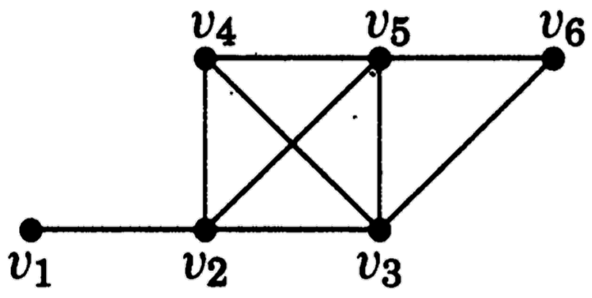
- Roots for the chromatic polynomials?
  Fundamental theorem of algebra
- A vertex of G is simplicial if its neighborhood in G induces a clique
- A simplicial elimination ordering is an ordering $v_n, \ldots, v_1$ for deletion of vertices s.t. each vertex v_i is a simplicial vertex of the graph reduced by $\{v_1, \ldots, v_i\}$
- Chromatic polynomials
  If we have colored $v_1, \ldots, v_{i-1}$, then there are $k - d(i)$ ways to color v_i where $d(i) = |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$. Thus

$$\chi(G;k) = \prod_{i=1}^{n} (k - d(i))$$

Nice factorization property!

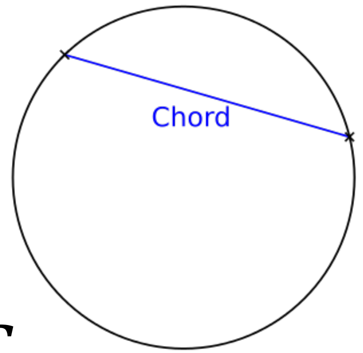
Examples

- In a tree, a simplicial elimination ordering is a successive deletion of leaves
  - Another proof for $\chi(T;k) = k(k-1)^{n-1}$
- Example (5.3.13, W) $v_6, \dots, v_1$ is a simplicial elimination ordering. The values $d(i)$ are 0,1,1,2,3,2. Thus the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$



- Exercise (Ex 5.3.19, W) There exists some graph without simplicial elimination ordering but has a nice factorization form for chromatic polynomial
  - The existence of simplicial elimination ordering is a sufficient condition for the chromatic polynomial having all real roots, but not necessary

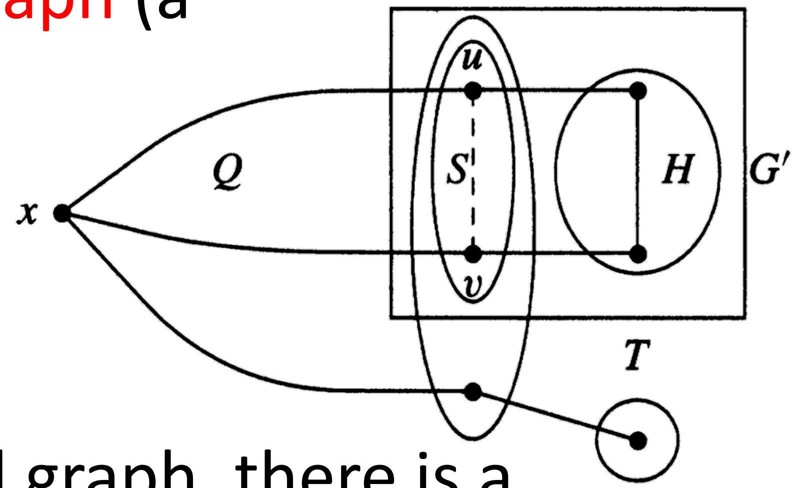
Chordal graphs



- A chord of a cycle C is an edge not in C whose endpoints lie in C
- A chordless cycle in G is a cycle of length at least 4 that has no chord
- Theorem (5.3.17, W; Dirac 1961) A simple graph has a simplicial elimination ordering $\Leftrightarrow$ it is a chordal graph (a simple graph without chordless cycle)
- TONCAS!
- Further

$$\chi(C_n; k) = (k-1)^n + (-1)^n (k-1)$$

  does not have a degree-1 decomposition
- Lemma (5.3.16, W) For every vertex x in a chordal graph, there is a simplicial vertex of G among the vertices farthest from x



Lecture 8: Planarity

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Motivation

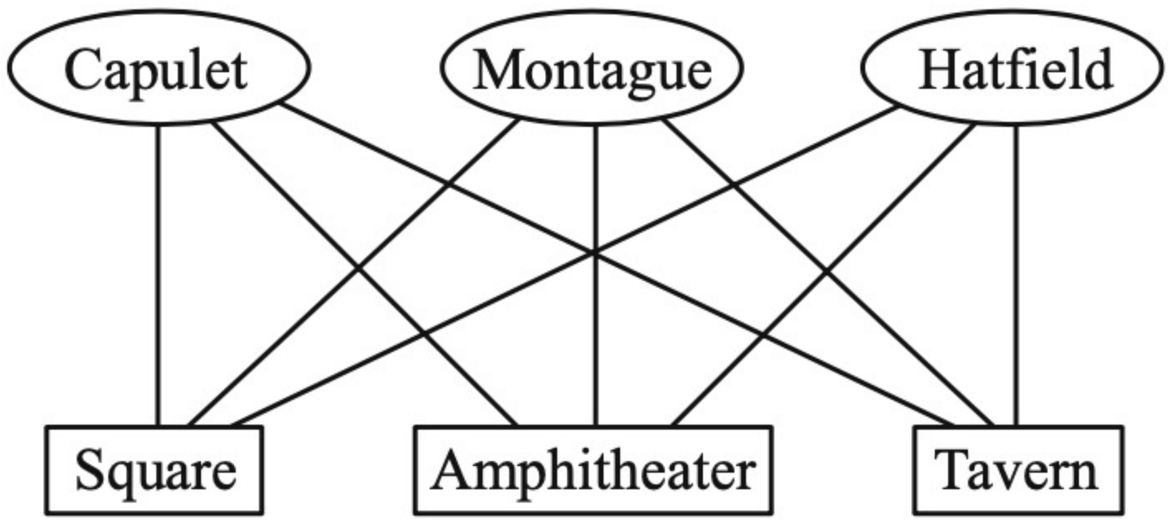


FIGURE 1.72. Original routes.

Definition and examples

- A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If G has no such representation, G is called nonplanar
- A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G

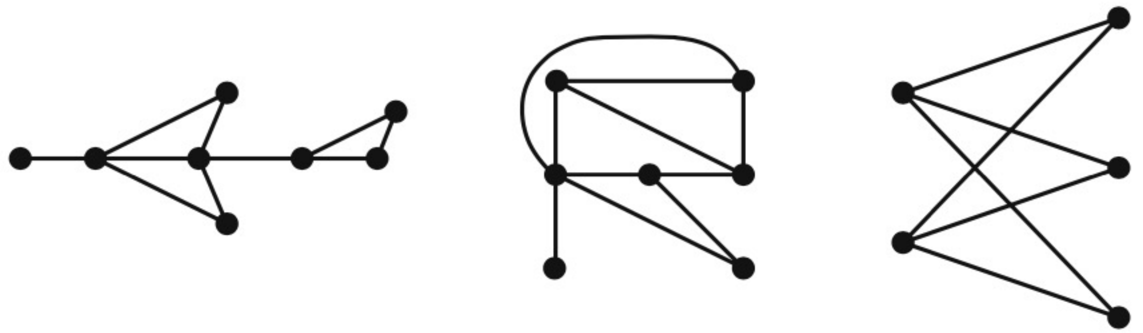
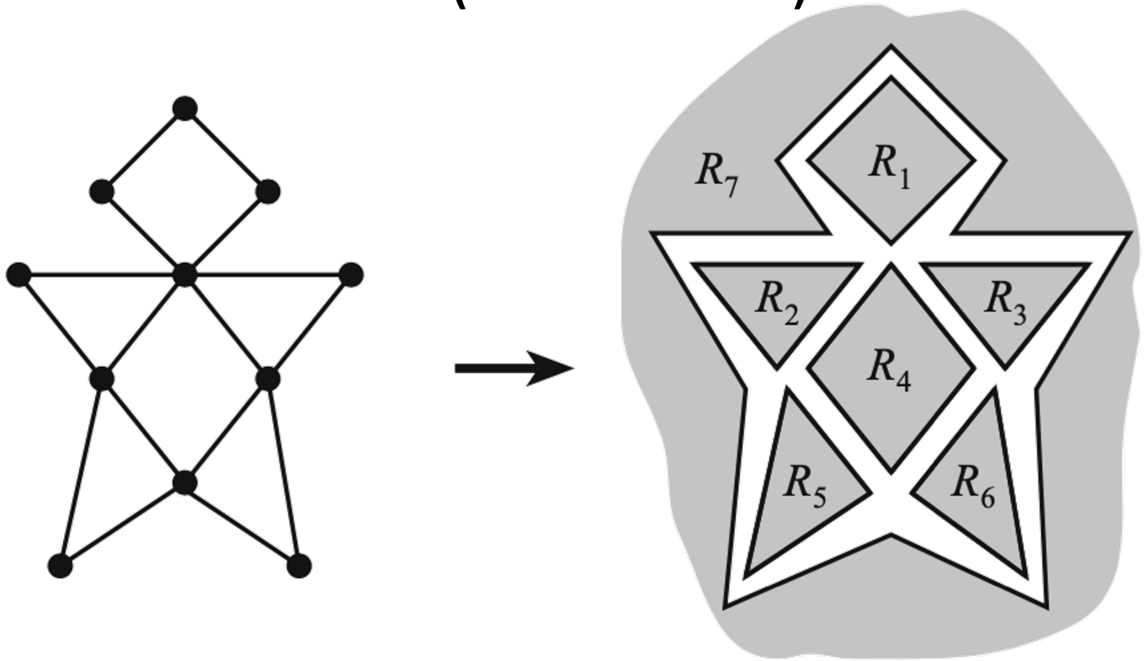


FIGURE 1.73. Examples of planar graphs.

Face

- Given a planar representation of a graph G, a face is a maximal region (polygonal open set) of the plane in which any two points can be joined by a curve that does not intersect any part of G
- The face R_7 is called the outer (or exterior) face



Face - properties

- An edge can come into contact with either one or two faces
- Example:
  - Edge e_1 is only in contact with one face S_1
  - Edge e_2, e_3 are only in contact with S_2
  - Each of other edges is in contact with two faces
- An edge e bounds a face F if e comes into contact with F and with a face different from F
- The bounded degree $b(F)$ is the number of edges that bound the face
  - Example: $b(S_1) = b(S_3) = 3, b(S_2) = 6$

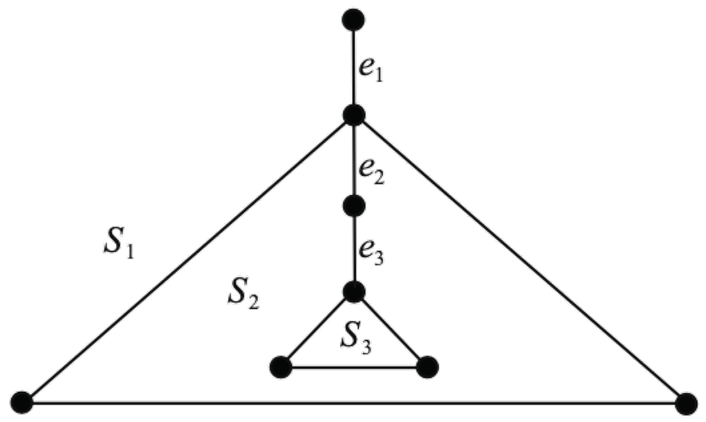


FIGURE 1.76. Edges e_1, e_2, and e_3 touch one face only.

Face - properties 2

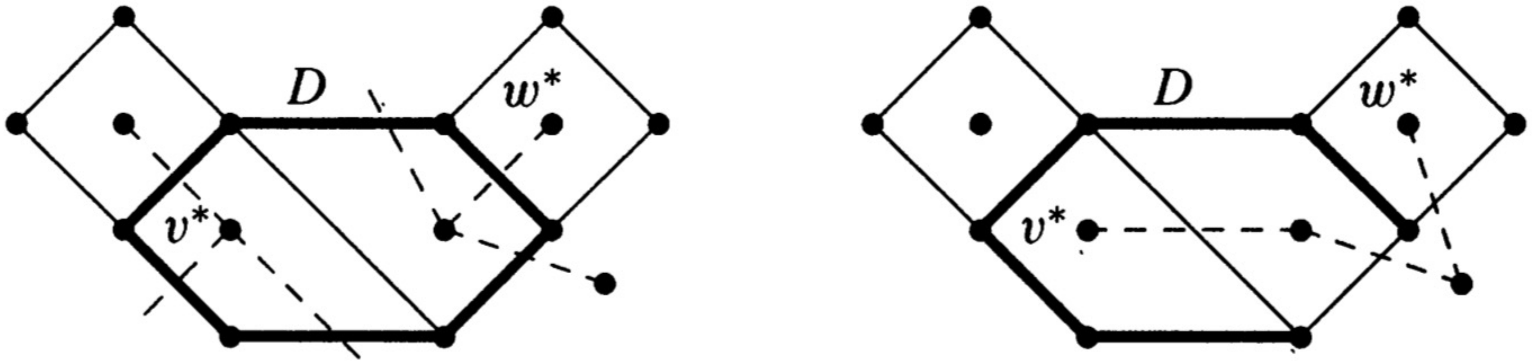
- The length of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face
- Proposition (6.1.13, W) If $l(F)$ denotes the length of face F in a plane graph G, then $2|E(G)| = \sum l(F_i)$
- Theorem (Restricted Jordan Curve Theorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary

Bond

- An edge cut may contain another edge cut
- Example: $K_{1,2}$ or star graphs
- A bond is a minimal nonempty edge cut
- Proposition (4.1.15, W) If G is a connected graph, then an edge cut F is a bond $\Leftrightarrow$ $G - F$ has exactly two components

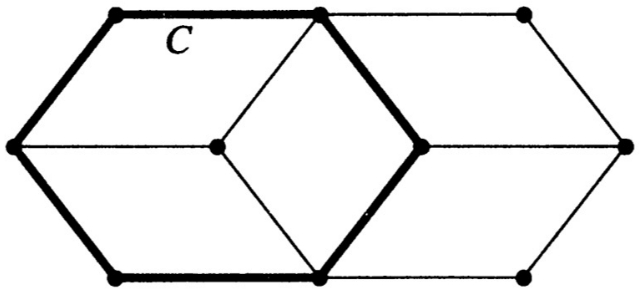
Dual graph

- The dual graph G^* of a plane graph G is a plane graph whose vertices are faces of G and edges are those contacting two faces
- Theorem (6.1.14, W) Edges in a plane graph G form a cycle in $G \Leftrightarrow$ the corresponding dual edges form a bond in G^*



Dual graph of bipartite graph

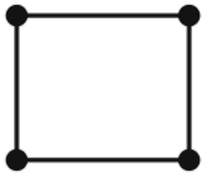
- Theorem (6.1.16, W) TFAE for a plane graph G
  - (a) G is bipartite
  - (b) Every face of G has even length
  - (c) The dual graph G^* is Eulerian



Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

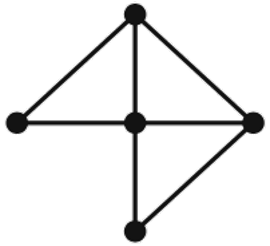
The relationship between numbers of vertices, edges and faces

- The number of vertices n
- The number of edges m
- The number of faces f

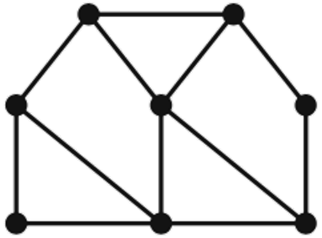


$n = 4$
$m = 4$
$f = 2$

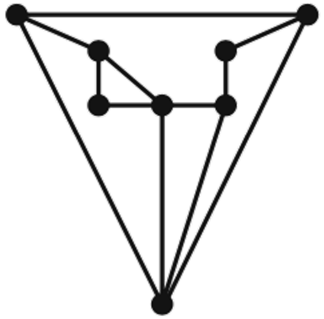
$n = 7$
$m = 9$
$f = 4$



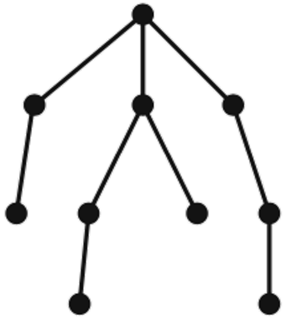
$n = 5$
$m = 7$
$f = 4$



$n = 8$
$m = 12$
$f = 6$



$n = 8$
$m = 12$
$f = 6$



$n = 10$
$m = 9$
$f = 1$

Euler's formula

- Theorem (1.31, H; 6.1.21, W; Euler 1758) If G is a connected planar graph with n vertices, m edges, and f faces, then
$$n - m + f = 2$$
  - Need Lemma: (Ex4, S1.5.1, H) Every tree is planar
- (Ex6, S1.5.2, H) Let G be a planar graph with k components. Then
$$n - m + f = k + 1$$

$K_{3,3}$ is nonplanar

- Theorem (1.32, H) $K_{3,3}$ is nonplanar

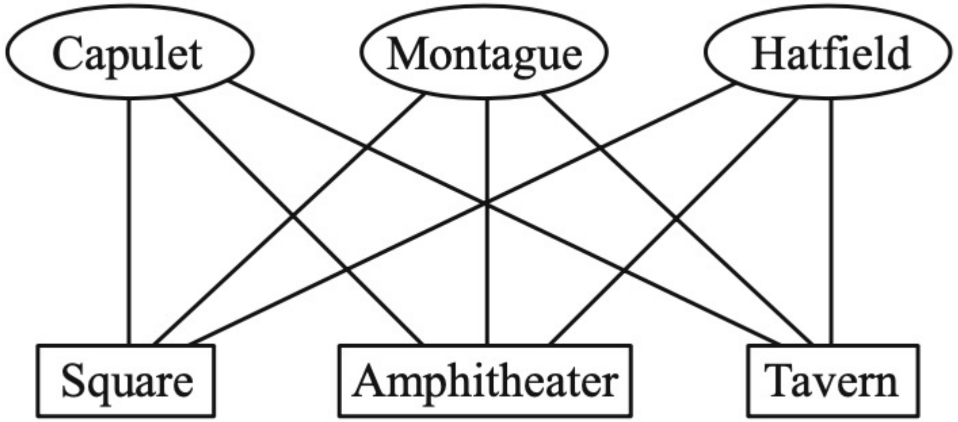


FIGURE 1.72. Original routes.

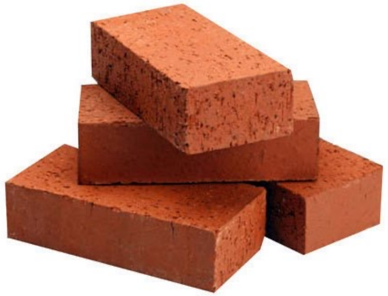
Upper bound for m

- Theorem (1.33, H; 6.1.23, W) If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds, then every face is bounded by 3 edges. In this case, G is maximal
- (Ex4, S1.5.2, H) Let G be a connected, planar, K_3-free graph of order $n \geq 3$. Then G has no more than $2n - 4$ edges
- Corollary (1.34, H) K_5 is nonplanar
- Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$
- (Ex5, S1.5.2, H) If G is bipartite planar graph, then $\delta(G) < 4$

Polyhedra

(Convex) Polyhedra 多面体

- A polyhedron is a solid that is bounded by flat surfaces

Polyhedra are planar

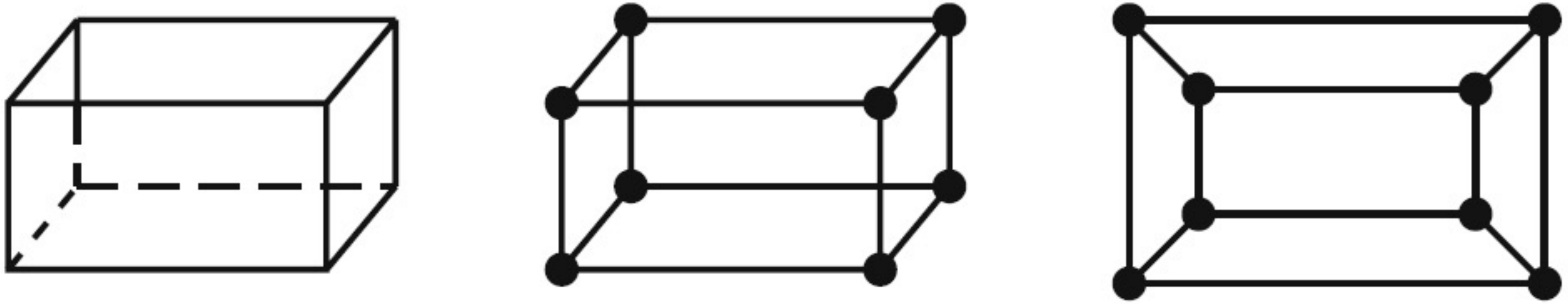


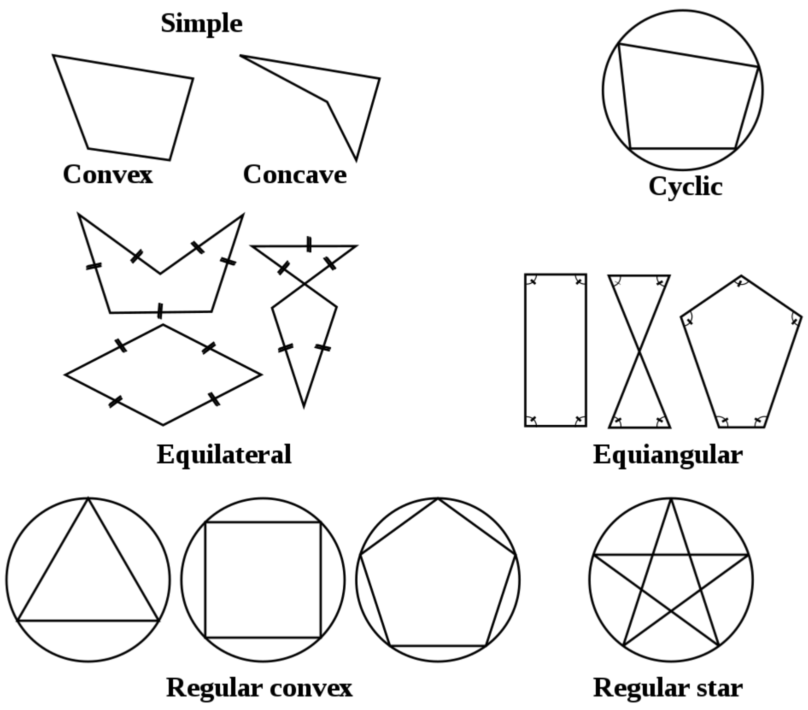
FIGURE 1.81. A polyhedron and its graph.

Properties

- Theorem (1.36, H) If a polyhedron has n vertices, m edges, and f faces, then
$$n - m + f = 2$$
- Given a polyhedron P, define
$$\rho(P) = \min\{l(F) : F \text{ is a face of } P\}$$
- Theorem (1.37, H) For all polyhedron P, $3 \leq \rho(P) \leq 5$

Regular polyhedron 正多面体

- A regular polygon is one that is equilateral and equiangular
  正多边形(cycle)，等边、等角
- A polyhedron is regular if its faces are mutually congruent, regular polygons and if the number of faces meeting at a vertex is the same for every vertex
  正多面体
  面是相互全等的、正多边形、点的度数相等



Regular polyhedron 正多面体

- Theorem (1.38, H; 6.1.28, W) There are exactly five regular polyhedral
- 正四面体
- 立方体（正六面体）
- 正八面体
- 正十二面体
- 正二十面体

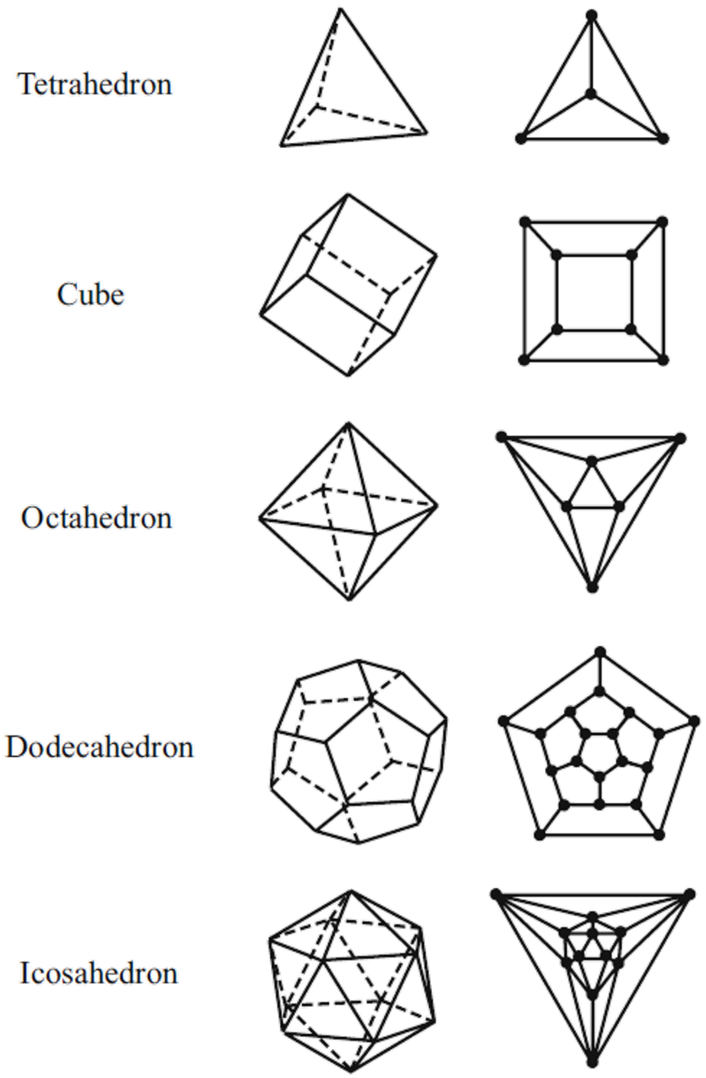


FIGURE 1.82. The five regular polyhedra and their graphical representations.

Kuratowski’s Theorem

Kuratowski's Theorem

- Theorem (1.39, H; Ex1, S1.5.4, H) A graph G is planar $\Longleftrightarrow$ every subdivision of G is planar

- Theorem (1.40, H; Kuratowski 1930) A graph is planar $\Longleftrightarrow$ it contains no subdivision of $K_{3,3}$ or K_5

The Four Color Problem

The Four Color Problem

- Q: Is it true that the countries on any given map can be colored with four or fewer colors in such a way that adjacent countries are colored differently?
- Theorem (Four Color Theorem) Every planar graph is 4-colorable
- Theorem (Five Color Theorem) (1.47, H; 6.3.1, W) Every planar graph is 5-colorable

> Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$

- Exercise (Ex5, S1.6.3, H) Where does the proof go wrong for four colors?

Lecture 9: Ramsey Theory

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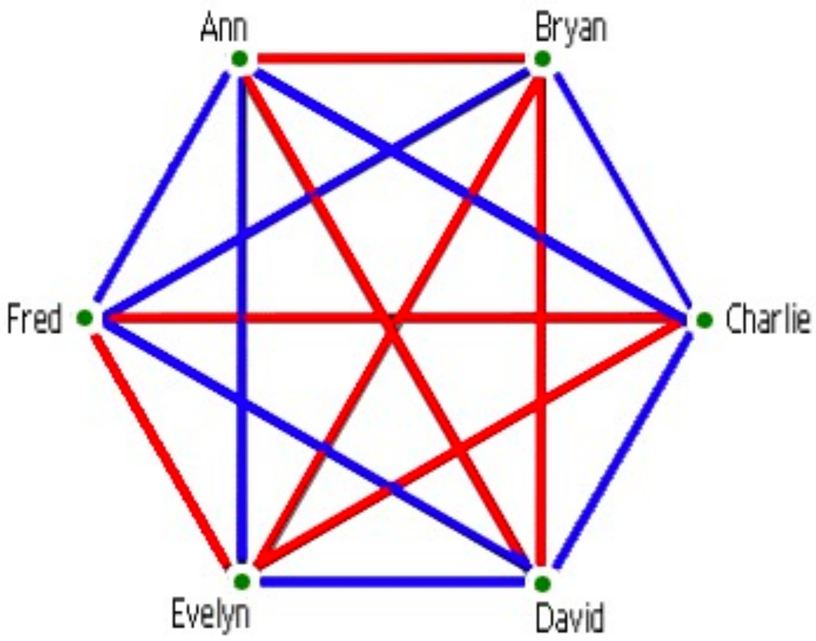
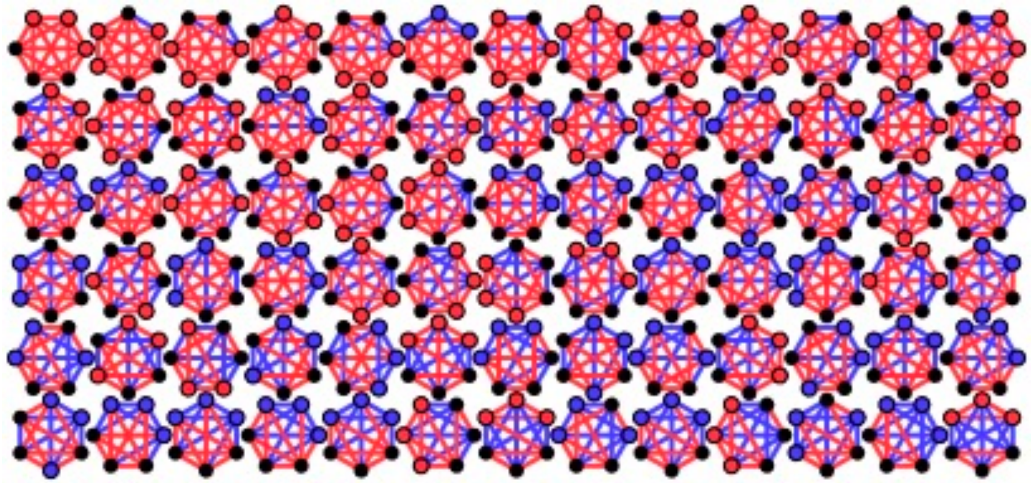
https://shuaili8.github.io

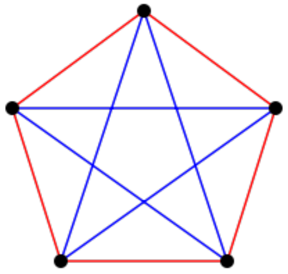
https://shuaili8.github.io/Teaching/CS445/index.html

The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?





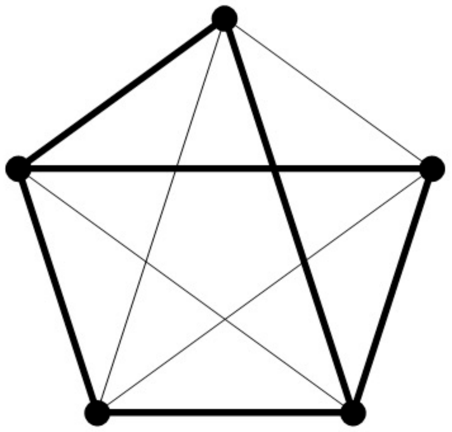


R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

https://plus.maths.org/content/friends-and-strangers
Wikipedia

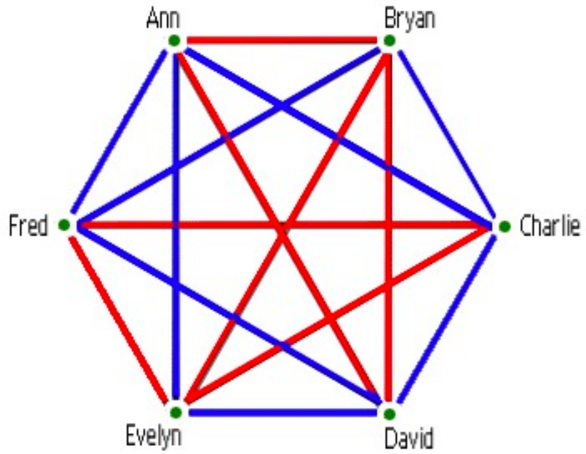
(classical) Ramsey number

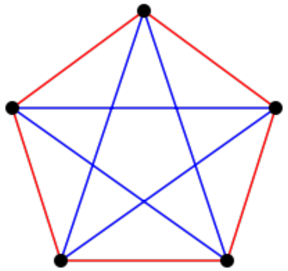
- A 2-coloring of the edges of a graph G is any assignment of one of two colors of each of the edges of G
- Let p and q be positive integers. The (classical) Ramsey number associated with these integers, denoted by $R(p, q)$, is defined to be the smallest integer n such that every 2-coloring of the edges of K_n either contains a red K_p or a blue K_q as a subgraph
- It is a typical problem of extremal graph theory



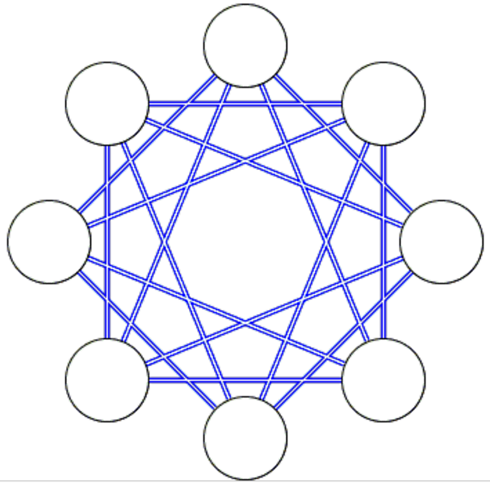
Examples

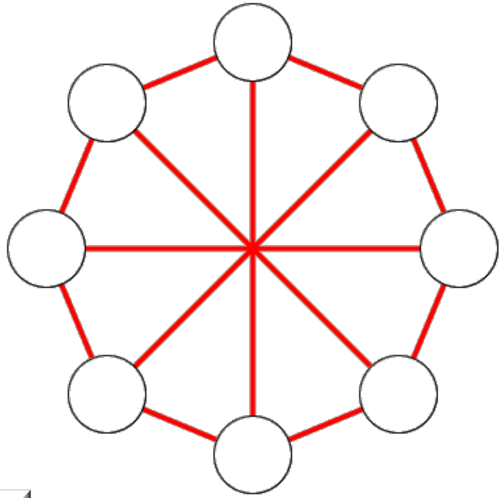
- $R(1,3) = 1$
- (Ex2, S1.8.1, H) $R(1,k) = 1$
- $R(2,4) = 4$
- (Ex3, S1.8.1, H) $R(2,k) = k$
- Theorem (1.61, H; 8.3.1, 8.3.9, W) $R(3,3) = 6$





Examples (cont.)





- Theorem (1.62, H; 8.3.10, W) $R(3,4) = 9$

Theorem A finite graph G has an even number of vertices with odd degree

- (Ex4, S1.8.1, H) $R(p,q) = R(q,p)$

Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[10]	36–41	49–61	59[14]–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149[14]–442
6						**102–165**	115[14]–298	134[14]–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090
9									**565–6588**	581–12677
10										**798–23556**

Bounds on Ramsey numbers

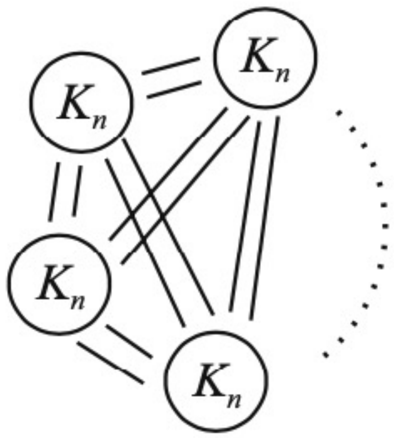
- Theorem (1.64, H; 2.28, H; 8.3.11, W) If $q \geq 2, q \geq 2$, then
$$R(p,q) \leq R(p-1,q) + R(p,q-1)$$
Furthermore, if both terms on the RHS are even, then the inequality is strict

> Theorem A finite graph G has an even number of vertices with odd degree

- Theorem (1.63, H; 2.29, H) $R(p,q) \leq \binom{p+q-2}{p-1}$
- Theorem (1.65, H) For integer $q \geq 3$, $R(3,q) \leq \frac{q^2+3}{2}$
- Theorem (1.66, H; 8.3.12, W; Erdős and Szekeres 1935)
If $p \geq 3$, $R(p,p) > \lfloor 2^{p/2} \rfloor$

Graph Ramsey Theory

- Given two graphs G and H, define the graph Ramsey number $R(G,H)$ to be the smallest value of n such that any 2-coloring of the edges of K_n contains either a red copy of G or a blue copy of H
  - The classical Ramsey number $R(p,q)$ would in this context be written as $R(K_p,K_q)$
- Theorem (1.67, H) If G is a graph of order p and H is a graph of order q, then $R(G,H) \leq R(p,q)$
- Theorem (1.68, H) Suppose the order of the largest component of H is denoted as $C(H)$.
  Then $R(G,H) \geq (\chi(G)-1)(C(H)-1)+1$



Graph Ramsey Theory (cont.)

- Theorem (1.69, H; 8.3.14, W) $R(T_m, K_n) = (m-1)(n-1) + 1$

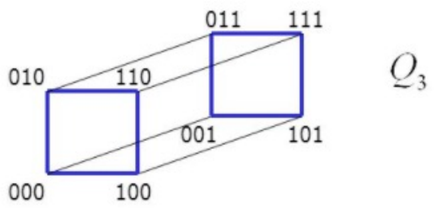
Theorem (1.45, H; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

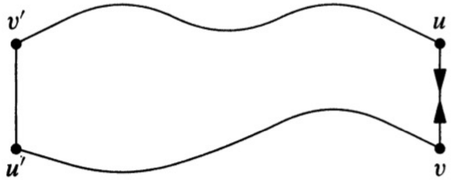
Proposition (5.2.13, W) Let G be a k-critical graph
(a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k-1$ colors all appear on $N(v)$
$\Rightarrow \delta(G) \geq k-1$

Theorem (1.16, H) Let T be a tree of order $k+1$ with k edges. Let G be a graph with $\delta(G) \geq k$. Then G contains T as a subgraph

More on pigeonhole principle



- Proposition (8.3.1, W) Among 6 people, it is possible to find 3 mutual acquaintances or 3 mutual non-acquaintances
  - $\Leftrightarrow$ For every simple graph with 6 vertices, there is a triangle in G or in $\bar{G}$
- Theorem (8.3.2, W) If T is a spanning tree of the k-dimensional cube Q_k, then there is an edge of Q_k outside T whose addition to T creates a cycle of length at least $2k$



T is a tree of order n
$\Leftrightarrow$ Any two vertices of T are linked by a unique path in T

  - $\Rightarrow$ Every spanning tree of Q_k has diameter at least $2k-1$

More on pigeonhole principle 2

- Theorem (8.3.3, W; Erdős–Szekeres 1935) Every list of $\geq n^2 + 1$ distinct numbers has a monotone sublist of length $\geq n + 1$
  - Generalization. $(r-1)(s-1)+1$
- Theorem (8.3.4, W; Graham-Kleitman 1973) In every labeling of $E(K_n)$ using distinct integers, there is a walk of length at least $n - 1$ along which the labels strictly increase